

Invariants of third-order ordinary differential equations $y''' = f(x, y, y', y'')$ via fiber preserving transformations

Ahmad Y. Al-Dweik*, M. T. Mustafa**, H. Azad* and F. M. Mahomed***,¹

*Department of Mathematics & Statistics, King Fahd University of Petroleum and Minerals, Dhahran 31261, Saudi Arabia

**Department of Mathematics, Statistics and Physics, Qatar University, Doha, 2713, State of Qatar

***School of Computational and Applied Mathematics, DST-NRF Centre of Excellence in Mathematical and Statistical Sciences; Differential Equations, Continuum Mechanics and Applications, University of the Witwatersrand, Johannesburg, Wits 2050, South Africa and

¹School of Mathematics and Statistics, University of New South Wales, Sydney NSW 2052 Australia

aydweik@kfupm.edu.sa, tahir.mustafa@qu.edu.qa, hassanaz@kfupm.edu.sa and Fazal.Mahomed@wits.ac.za

Abstract

Bagderina [1] solved the equivalence problem for scalar third-order ordinary differential equations (ODEs), quadratic in the second-order derivative, via point transformations. However, the question is open for the general class $y''' = f(x, y, y', y'')$ which is not quadratic in the second-order derivative. We utilize Lie's infinitesimal method to study the differential invariants of this general class under pseudo-group

¹Author FM is visiting professorial fellow at UNSW, Sydney for 2014

of fiber preserving equivalence transformations $\bar{x} = \phi(x), \bar{y} = \psi(x, y)$. As a result, all third-order differential invariants of this group and the invariant differentiation operators are determined. This leads to simple necessary explicit conditions for a third-order ODE to be equivalent to the respective canonical form under the considered group of transformations. Applications motivated by the literature are presented.

Keywords: Lie's infinitesimal method, differential invariants, third order ODEs, equivalence problem, fiber preserving transformations, normal forms, Lie symmetries.

1 Introduction

Differential invariants play a significant role in a broad range of problems arising in differential geometry, differential equations, mathematical physics and applications. For instance, differential invariants have been particularly useful in dealing with the equivalence problem for geometric structures [3], classification of invariant differential equations and invariant variational problems [4, 5, 6, 7], integration of ordinary differential equations (ODEs) [5, 6], equivalence and symmetry properties of solutions [3], and in the construction of particular solutions to systems of partial differential equations (PDEs) [6, 8, 9, 10].

Lie [11] showed that every invariant system of differential equations [12] and variational problem [14] can be directly expressed in terms of differential invariants. Along with an illustration [12] of how differential invariants could be used to integrate ODEs, Lie succeeded in completely classifying all the differential invariants for all possible finite-dimensional Lie groups of point transformations in the complex plane.

Tressé [15] and Ovsiannikov [6] gave prominence to Lie's preliminary results on invariant differentiations and the existence of finite bases of differential invariants. For the general theory of differential invariants of Lie groups including algorithms of construction of differential invariants, the interested reader is referred to [3, 6]. Ibragimov [16, 18] developed a simple method for constructing invariants of families of linear and nonlinear differential equations admitting infinite equivalence transformation groups. Lie's infinitesimal method was applied to solve the equivalence problem for several linear and nonlinear differential equations [19, 20, 21, 22, 23, 24, 25, 26, 27]. Cartan's equivalence method [3, 28] is another systematic approach to solve the equivalence problem for differential equations. The linearization problem is a particular case of the equivalence problem.

Linearization criteria for a third-order ODE which are at most cubic in the second-order derivative

$$y''' = a(x, y, y')y'''^3 + b(x, y, y')y''^2 + c(x, y, y')y'' + d(x, y, y') \quad (1.1)$$

have been obtained in [29, 30] by Cartan's method and then in [23] by the direct approach. Lie [13] in fact was the first to note that the third-order ODE connected via contact transformations to the simplest linear third-order ODE is at most cubic in the second-order derivative of the form given above in (1.1). Recently, Bagderina [2] presented the basis of differential invariants under the group of contact transformations for the family of ODEs at most cubic in the second-order derivative (1.1) by using Lie's method. She also provided the operators of invariant differentiations.

By using Lie's infinitesimal method, Bagderina [1] solved the equivalence problem of third-order ODEs which are at most quadratic in the second-order derivative

$$y''' = a(x, y, y')y''^2 + b(x, y, y')y'' + c(x, y, y') \quad (1.2)$$

with respect to the group of point equivalence transformations

$$\bar{x} = \phi(x, y), \bar{y} = \psi(x, y). \quad (1.3)$$

As an extension, in this paper, we use Lie's infinitesimal method to study the differential invariants of the third-order ODEs

$$y''' = f(x, y, y', y''), \quad (1.4)$$

which are not quadratic in the second-order derivative, under pseudo-group of fiber preserving equivalence transformations

$$\bar{x} = \phi(x), \bar{y} = \psi(x, y). \quad (1.5)$$

The structure of the paper is as follows. In the next section, we give a short description of Lie's infinitesimal method to find the differential invariants and invariant differentiation operators of the class of ODEs (1.4) with respect to the general group of point equivalence transformations $\bar{x} = \phi(x, y), \bar{y} = \psi(x, y)$. In Section 3, using the methods described in Section 2, first, we recover the infinitesimal point equivalence transformations. Then we find the third-order differential invariants and invariant differentiation operators of the class of ODEs (1.4), which are not quadratic in the second-order derivative, under two subgroups of the general group of point equivalence transformations. In Section 4, we provide illustrative examples of equations not quadratic in the second-order derivative taken from [17, 33, 34]. This is motivated by studies of this more general class for symmetry properties in [17], exact solutions for certain third-order ODEs in [33], linearization and equivalence under contact transformation for the class (1.1) in [23, 30, 2] as well as in the definition of certain Einstein-Weyl geometry of hyper-CR type which is of recent interest in physics [34]. Here we consider examples that are equivalent under the pseudo-group of fiber preserving equivalence transformations. Another motivation for the examples considered is that third-order ODEs possessing the Painlevé property for polynomial in its lower order derivatives were also investigated, see e.g. [31, 32]. Finally the conclusion is presented.

Throughout this paper, we use the notation $A = [a_1, a_2, \dots, a_n]$ to express any differential

operator $A = \sum_{j=1}^n a_j \frac{\partial}{\partial b_j}$. Also, we denote y', y'' by p, q , respectively.

2 Lie's infinitesimal method

In this section, we briefly describe the Lie method used to derive differential invariants using point equivalence transformations.

Consider the k th-order system of PDEs of n independent variables $x = (x^1, x^2, \dots, x^n)$ and m dependent variables $u = (u^1, u^2, \dots, u^m)$

$$E_\alpha(x, u, \dots, u_{(k)}) = 0, \quad \alpha = 1, \dots, m, \quad (2.6)$$

where $u_{(1)}, u_{(2)}, \dots, u_{(k)}$ denote the collections of all first, second, ..., k th-order partial derivatives, i.e., $u_i^\alpha = D_i(u^\alpha)$, $u_{ij}^\alpha = D_j D_i(u^\alpha), \dots$, respectively, in which the total differentiation operator with respect to x^i is

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \dots, \quad i = 1, \dots, n, \quad (2.7)$$

with the summation convention adopted for repeated indices.

Definition 2.1. *The Lie-Bäcklund operator is*

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} \quad \xi^i, \eta^\alpha \in A, \quad (2.8)$$

where A is the space of *differential functions*.

The operator (2.8) is an abbreviated form of the infinite formal sum

$$\begin{aligned} X^{(s)} &= \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{s \geq 1} \zeta_{i_1 i_2 \dots i_s}^\alpha \frac{\partial}{\partial u_{i_1 i_2 \dots i_s}^\alpha}, \\ &= \xi^i D_i + W^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{s \geq 1} D_{i_1} \dots D_{i_s} (W^\alpha) \frac{\partial}{\partial u_{i_1 i_2 \dots i_s}^\alpha}, \end{aligned} \quad (2.9)$$

where the additional coefficients are determined uniquely by the prolongation formulae

$$\begin{aligned}\zeta_i^\alpha &= D_i(\eta^\alpha) - u_j^\alpha D_i(\xi^j) = D_i(W^\alpha) + \xi^j u_{ij}^\alpha, \\ \zeta_{i_1 \dots i_s}^\alpha &= D_{i_s}(\zeta_{i_1 \dots i_{s-1}}^\alpha) - u_{j i_1 \dots i_{s-1}}^\alpha D_{i_s}(\xi^j) = D_{i_1} \dots D_{i_s}(W^\alpha) + \xi^j u_{j i_1 \dots i_s}^\alpha, \quad s > 1,\end{aligned}\tag{2.10}$$

in which W^α is the *Lie characteristic function*

$$W^\alpha = \eta^\alpha - \xi^j u_j^\alpha.\tag{2.11}$$

Definition 2.2. *The point equivalence transformation of a class of PDEs (2.6) is an invertible transformation of the independent and dependent variables of the form*

$$\bar{x} = \phi(x, u), \bar{u} = \psi(x, u),\tag{2.12}$$

that maps every equation of the class into an equation of the same family, viz.

$$E_\alpha(\bar{x}, \bar{u}, \dots, \bar{u}_{(k)}) = 0, \quad \alpha = 1, \dots, m.\tag{2.13}$$

In order to describe Lie's infinitesimal method for deriving differential invariants using point equivalence transformations, we use as example the class of equations (1.4). It is well-known that the point equivalence transformation

$$\bar{x} = \phi(x, y), \bar{y} = \psi(x, y),\tag{2.14}$$

maps (1.4) into the same family, viz.

$$\bar{y}''' = \bar{f}(\bar{x}, \bar{y}, \bar{y}', \bar{y}''),\tag{2.15}$$

for arbitrary functions $\phi(x, y)$ and $\psi(x, y)$, where \bar{f} , in general, can be different from the original function f . The set of all equivalence transformations forms a group denoted by \mathcal{E} .

The standard procedure for Lie's infinitesimal invariance criterion [6] is implemented in the next section to recover the continuous group of point equivalence transformations

(2.14) for the class of third-order ODEs (1.4) with the corresponding infinitesimal point equivalence transformation operator

$$Y = \xi(x, y)D_x + W\partial_y + D_x(W)\partial_p + D_x^2(W)\partial_q + \mu(x, y, p, q, f)\partial_f, \quad (2.16)$$

where $\xi(x, y)$ and $\eta(x, y)$ are arbitrary functions obtained from

$$\bar{x} = x + \epsilon\xi(x, y) + O(\epsilon^2) = \phi(x, y), \quad (2.17)$$

$$\bar{y} = y + \epsilon\eta(x, y) + O(\epsilon^2) = \psi(x, y), \quad (2.18)$$

and

$$\mu = \dot{D}_x^3(W) + \xi(x, y)\dot{D}_x f, \quad (2.19)$$

with $W = \eta - \xi p$ and $\dot{D}_x = \frac{\partial}{\partial x} + p\frac{\partial}{\partial y} + q\frac{\partial}{\partial p} + f\frac{\partial}{\partial q}$.

Definition 2.3. *An invariant of a class of third-order ODEs (1.4) is a function of the form*

$$J = J(x, y, p, q, f), \quad (2.20)$$

which is invariant under the equivalence transformation (2.14).

Definition 2.4. *A differential invariant of order s of a class of third-order ODEs (1.4) is a function of the form*

$$J = J(x, y, p, q, f, f_{(1)}, f_{(2)}, \dots, f_{(s)}), \quad (2.21)$$

which is invariant under the equivalence transformation (2.14) where $f_{(1)}, f_{(2)}, \dots, f_{(s)}$ denote the collections of all first, second, ..., s -th-order partial derivatives.

Definition 2.5. *An invariant system of order s of a class of third-order ODEs (1.4) is the system of the form $E_\alpha(x, y, p, q, f, f_{(1)}, f_{(2)}, \dots, f_{(s)}) = 0$, $\alpha = 1, \dots, m$ which satisfies the condition*

$$Y^{(s)}E_\alpha(x, y, p, q, f, f_{(1)}, f_{(2)}, \dots, f_{(s)}) = 0 \pmod{E_\alpha = 0, \alpha = 1, \dots, m}, \quad \alpha = 1, \dots, m. \quad (2.22)$$

An invariant system with $\alpha = 1$ is called an invariant equation.

Now, according to the theory of invariants of infinite transformation groups [6], the invariant criterion

$$YJ(x, y, p, q, f) = 0, \quad (2.23)$$

should be split by means of the functions $\xi(x, y)$ and $\eta(x, y)$ and their derivatives. This gives rise to a homogeneous linear system of PDEs whose solution gives the required invariants.

It should be noted that since the generator Y contains arbitrary functions $\xi(x, y)$ and $\eta(x, y)$, the corresponding identity (2.23) leads to m linear PDEs for J , where m is the number of the arbitrary functions and their derivatives that appear in Y . We point out that these m PDEs are not necessarily linearly independent.

In order to determine the differential invariants of order s , we need to calculate the prolongations of the operator Y using (2.9) by considering f as a dependent variable and the variables x, y, p, q as independent variables:

$$Y^{(s)} = Y(x)\tilde{D}_x + Y(y)\tilde{D}_y + Y(p)\tilde{D}_p + Y(q)\tilde{D}_q + \tilde{W}\frac{\partial}{\partial f} + \sum_{s \geq 1} \tilde{D}_{i_1} \dots \tilde{D}_{i_s}(\tilde{W}) \frac{\partial}{\partial f_{i_1 i_2 \dots i_s}},$$

$$i_1, i_2, \dots, i_s \in \{x, y, p, q\}, \quad (2.24)$$

where

$$\tilde{D}_k = \partial_k + f_k \partial_f + f_{ki} \partial_{f_i} + f_{kij} \partial_{f_{ij}} + \dots, \quad i, j, k \in \{x, y, p, q\}. \quad (2.25)$$

in which \tilde{W} is the *Lie characteristic function*

$$\tilde{W} = \mu - Y(x)f_x - Y(y)f_y - Y(p)f_p - Y(q)f_q. \quad (2.26)$$

The differential invariants are determined by the equations

$$Y^{(s)}J(x, y, p, q, f, f_{(1)}, f_{(2)}, \dots, f_{(s)}) = 0. \quad (2.27)$$

It should be noted that since the generator $Y^{(s)}$ contains arbitrary functions $\xi(x, y)$ and $\eta(x, y)$, the corresponding identity (2.27) leads to m linear PDEs for J , where m is the number of the arbitrary functions and their derivatives that appear in $Y^{(s)}$.

For simplicity, from here on, we denote the derivative of $f(x, y, p, q)$ with respect to the independent variables x, y, p, q as f_1, f_2, f_3, f_4 . The same notation will be used for higher-order derivatives.

Now, in order to find all the third order differential invariants of the third-order ODE (1.4), one can solve the invariant criterion (2.27) with $s = 3$. However, for compactness of the derived differential invariants, one can replace any partial derivative with respect to x by the total derivative with respect to x . So, we need to solve the following invariant criterion

$$Y^{(3)}J(x, y, p, q, f, f_2, f_3, f_4, f_{2,2}, f_{2,3}, f_{2,4}, f_{3,3}, f_{3,4}, f_{4,4}, f_{2,2,2}, f_{2,2,3}, f_{2,2,4}, f_{2,3,3}, f_{2,3,4}, f_{2,4,4}, f_{3,3,3}, f_{3,3,4}, f_{3,4,4}, f_{4,4,4}, d_{1,1}, d_{1,2}, d_{1,3}, d_{1,4}, d_{1,5}, d_{1,6}, d_{1,7}, d_{1,8}, d_{1,9}, d_{1,10}, d_{2,1}, d_{2,2}, d_{2,3}, d_{2,4}, d_{3,1}) = 0, \quad (2.28)$$

by prolonging the infinitesimal operator $Y^{(3)}$ to the variables $d_{i,j}$ through the infinitesimals $Y^{(3)}d_{i,j}$, where

$$\begin{aligned} d_{1,1} &= \dot{D}_x f, d_{1,2} = \dot{D}_x f_2, d_{1,3} = \dot{D}_x f_3, d_{1,4} = \dot{D}_x f_4, d_{1,5} = \dot{D}_x f_{2,2}, \\ d_{1,6} &= \dot{D}_x f_{2,3}, d_{1,7} = \dot{D}_x f_{2,4}, d_{1,8} = \dot{D}_x f_{3,3}, d_{1,9} = \dot{D}_x f_{3,4}, d_{1,10} = \dot{D}_x f_{4,4}, \\ d_{2,1} &= \dot{D}_x^2 f, d_{2,2} = \dot{D}_x^2 f_2, d_{2,3} = \dot{D}_x^2 f_3, d_{2,4} = \dot{D}_x^2 f_4, d_{3,1} = \dot{D}_x^3 f. \end{aligned} \quad (2.29)$$

Definition 2.6. An invariant differentiation operator of a class of third-order ODEs (1.4) is a differential operator \mathcal{D} which satisfies that if I is a differential invariant of ODE (1.4), then $\mathcal{D}I$ is its differential invariant too.

As it is shown in [6], the number of independent invariant differentiation operators \mathcal{D} equals the number of independent variables x, y, p and q . The invariant differentiation operators \mathcal{D} should take the form

$$\mathcal{D} = K\tilde{D}_x + L\tilde{D}_y + M\tilde{D}_p + N\tilde{D}_q, \quad (2.30)$$

with the coordinates K, L, M and N satisfying the non-homogeneous linear system

$$Y^{(3)}K = \mathcal{D}(Y(x)), \quad Y^{(3)}L = \mathcal{D}(Y(y)), \quad Y^{(3)}M = \mathcal{D}(Y(p)), \quad Y^{(3)}N = \mathcal{D}(Y(q)), \quad (2.31)$$

where K, L, M and N are functions of the following variables

$$x, y, p, q, f, f_2, f_3, f_4, f_{2,2}, f_{2,3}, f_{2,4}, f_{3,3}, f_{3,4}, f_{4,4}, f_{2,2,2}, f_{2,2,3}, f_{2,2,4}, f_{2,3,3}, f_{2,3,4}, f_{2,4,4}, f_{3,3,3}, f_{3,3,4}, f_{3,4,4}, f_{4,4,4}, d_{1,1}, d_{1,2}, d_{1,3}, d_{1,4}, d_{1,5}, d_{1,6}, d_{1,7}, d_{1,8}, d_{1,9}, d_{1,10}, d_{2,1}, d_{2,2}, d_{2,3}, d_{2,4}, d_{3,1}. \quad (2.32)$$

In reality, the general solution of the system (2.31) gives both the differential invariants and the differential operators. This general solution can be found by prolonging the infinitesimal operator $Y^{(3)}$ to the variables K, L, M and N through the infinitesimals $Y^{(3)}K, Y^{(3)}L, Y^{(3)}M$ and $Y^{(3)}N$ respectively. Then solving the invariant criterion

$$Y^{(3)}J(x, y, p, q, f, f_2, f_3, f_4, f_{2,2}, f_{2,3}, f_{2,4}, f_{3,3}, f_{3,4}, f_{4,4}, f_{2,2,2}, f_{2,2,3}, f_{2,2,4}, f_{2,3,3}, f_{2,3,4}, f_{2,4,4}, f_{3,3,3}, f_{3,3,4}, f_{3,4,4}, f_{4,4,4}, d_{1,1}, d_{1,2}, d_{1,3}, d_{1,4}, d_{1,5}, d_{1,6}, d_{1,7}, d_{1,8}, d_{1,9}, d_{1,10}, d_{2,1}, d_{2,2}, d_{2,3}, d_{2,4}, d_{3,1}, K, L, M, N) = 0, \quad (2.33)$$

gives the implicit solution of the variables K, L, M and N with the differential invariants.

In this paper, we are interested in finding the third-order differential invariants and differential operators of the general class $y''' = f(x, y, y', y'')$ under a subgroup of point transformations (2.14), namely the fiber preserving transformations $x = \phi(x), y = \psi(x, y)$. So, according to the theory of invariants of infinite transformation groups [6], the invariant criterion (2.33) should be split by the functions $\xi(x)$ and $\eta(x, y)$ and their derivatives. This gives rise to a homogeneous linear system of partial differential equations (PDEs):

$$X_i J = 0, i = 1 \dots 28, \quad T_i J = 0, i = 1 \dots 7, \quad (2.34)$$

where $X_i, i = 1 \dots 28$, are the differential operators corresponding to the coefficients of the following derivatives of $\eta(x, y)$ up to the sixth order in the invariant criterion

$$\eta, \eta_1, \eta_2, \eta_{1,1}, \eta_{1,2}, \eta_{2,2}, \eta_{1,1,1}, \eta_{1,1,2}, \eta_{1,2,2}, \eta_{2,2,2}, \eta_{1,1,1,1}, \eta_{1,1,1,2}, \eta_{1,1,2,2}, \eta_{1,2,2,2}, \eta_{2,2,2,2}, \eta_{1,1,1,1,1}, \eta_{1,1,1,1,2}, \eta_{1,1,1,2,2}, \eta_{1,1,2,2,2}, \eta_{1,2,2,2,2}, \eta_{2,2,2,2,2}, \eta_{1,1,1,1,1,1}, \eta_{1,1,1,1,1,2}, \eta_{1,1,1,1,2,2}, \eta_{1,1,2,2,2,2}, \eta_{1,2,2,2,2,2}, \eta_{2,2,2,2,2,2} \quad (2.35)$$

and $T_i, i = 1 \dots 7$, are the differential operators corresponding to the coefficients of the following derivatives of $\xi(x)$ up to the sixth order in the invariant criterion

$$\xi, \xi_1, \xi_{1,1}, \xi_{1,1,1}, \xi_{1,1,1,1}, \xi_{1,1,1,1,1}, \xi_{1,1,1,1,1,1}. \quad (2.36)$$

The expressions for the differential operators $X_i, i = 1 \dots 28$ and $T_i, i = 1 \dots 7$ are

therefore too large and these are given in the Appendix A, after relabeling the variables

$$x, y, p, q, f, f_2, f_3, f_4, f_{2,2}, f_{2,3}, f_{2,4}, f_{3,3}, f_{3,4}, f_{4,4}, f_{2,2,2}, f_{2,2,3}, f_{2,2,4}, f_{2,3,3}, f_{2,3,4}, f_{2,4,4}, f_{3,3,3}, f_{3,3,4}, f_{3,4,4}, f_{4,4,4}, d_{1,1}, d_{1,2}, d_{1,3}, d_{1,4}, d_{1,5}, d_{1,6}, d_{1,7}, d_{1,8}, d_{1,9}, d_{1,10}, d_{2,1}, d_{2,2}, d_{2,3}, d_{2,4}, d_{3,1}, K, L, M, N, \quad (2.37)$$

by the variables $z_i, i = 1 \dots 43$, respectively.

Functionally independent solutions of system (2.34) provide all independent differential invariants of $y''' = f(x, y, y', y'')$ up to the third order under the fiber preserving transformation, as well as an implicit solution of the variables K, L, M and N which yield the differential operators via (2.30) as explained in the next section.

The existence of the solutions for system (2.34) is proved by showing that the differential operators $X_i, i = 1 \dots 28$ and $T_i, i = 1 \dots 7$ form a Lie algebra \mathcal{L}_{35} [3, p.422, Theorem 14.1]. The nonzero commutators for the Lie algebra \mathcal{L}_{35} are given in the Appendix B, after relabeling the differential operators $X_i, i = 1 \dots 28$ and $T_i, i = 1 \dots 7$ by the operators $e_i, i = 1 \dots 35$, respectively.

Using Appendix B, it can be seen that the Lie algebra \mathcal{L}_{35} is solvable and has the chain of Lie subalgebras $0 = \mathcal{G}_0 \triangleleft \mathcal{G}_1 \triangleleft \mathcal{G}_2 \triangleleft \mathcal{G}_3 \triangleleft \mathcal{G}_4 \triangleleft \mathcal{G}_5 \triangleleft \mathcal{G}_6 \triangleleft \mathcal{G}_7 \triangleleft \mathcal{G}_8 = \mathcal{L}_{35}$ with each an ideal in the next where

$$\begin{aligned} \mathcal{G}_1 &= \{e_{22}, e_{23}, e_{24}, e_{25}, e_{26}, e_{27}, e_{28}\}, \\ \mathcal{G}_2 &= \mathcal{G}_1 \cup \{e_{16}, e_{17}, e_{18}, e_{19}, e_{20}, e_{21}\}, \\ \mathcal{G}_3 &= \mathcal{G}_2 \cup \{e_{11}, e_{12}, e_{13}, e_{14}, e_{15}\}, \\ \mathcal{G}_4 &= \mathcal{G}_3 \cup \{e_7, e_8, e_9, e_{10}\}, \\ \mathcal{G}_5 &= \mathcal{G}_4 \cup \{e_4, e_5, e_6\}, \\ \mathcal{G}_6 &= \mathcal{G}_5 \cup \{e_{33}, e_{34}, e_{35}\}, \\ \mathcal{G}_7 &= \mathcal{G}_6 \cup \{e_{31}, e_{32}\}, \\ \mathcal{G}_8 &= \mathcal{G}_7 \cup \{e_1, e_2, e_3, e_{29}, e_{30}\}. \end{aligned} \quad (2.38)$$

In the next section, we solve the system (2.34) by using the chain (2.38). In more detail,

as \mathcal{G}_1 is abelian, one can find its joint invariants by finding the invariants of its generators in any order. Since \mathcal{G}_1 is an ideal in \mathcal{G}_2 , the algebra \mathcal{G}_2 operates on the joint invariants of \mathcal{G}_1 . Moreover, as \mathcal{G}_1 is abelian, then the induced action of \mathcal{G}_2 on the joint invariants of \mathcal{G}_1 is also abelian. Continuing in this manner, we obtain the joint invariants of the full algebra.

The reader is further referred to [16] and [18, Section 10] for examples illustrating the infinitesimal method.

3 Third-order differential invariants and invariant equations for $y''' = f(x, y, y', y'')$

3.1 The infinitesimal point equivalence transformations

In order to find continuous group of equivalence transformations of the class (1.4) we consider the arbitrary function f that appears in our equation as a dependent variable and the variables $x, y, y' = p, y'' = q$ as independent variables and apply the Lie infinitesimal invariance criterion [6], that is we look for the infinitesimal ξ, η and μ of the equivalence operator Y :

$$Y = \xi(x, y)\partial_x + \eta(x, y)\partial_y + \mu(x, y, p, q, f)\partial_f, \quad (3.39)$$

such that its prolongation leaves the equation (1.4) invariant.

The prolongation of operator Y can be given using (2.9) as

$$Y = \xi(x, y)D_x + W\partial_y + D_x(W)\partial_p + D_x^2(W)\partial_q + D_x^3(W)\partial_{y'''} + \mu(x, y, p, q, f)\partial_f, \quad (3.40)$$

where

$$D_x = \frac{\partial}{\partial x} + p \frac{\partial}{\partial y} + q \frac{\partial}{\partial p} + y''' \frac{\partial}{\partial q} + y^{(4)} \frac{\partial}{\partial y'''} + \dots$$

is the operator of total derivative and $W = \eta(x, y) - \xi(x, y)p$ is the characteristic of infinitesimal operator $X = \xi(x, y)\partial_x + \eta(x, y)\partial_y$.

So, the Lie infinitesimal invariance criterion gives $\mu = \dot{D}_x^3(W) + \xi(x, y)\dot{D}_x f$ for arbitrary functions $\xi(x, y)$ and $\eta(x, y)$ where $\dot{D}_x = \frac{\partial}{\partial x} + p \frac{\partial}{\partial y} + q \frac{\partial}{\partial p} + f \frac{\partial}{\partial q}$.

Thus, equation (1.4) admits an infinite continuous group of equivalence transformations generated by the Lie algebra $\mathcal{L}_\mathcal{E}$ spanned by the following infinitesimal operators

$$U = \xi(x, y) \frac{\partial}{\partial x} - p D_x(\xi) \partial_p - (2q D_x(\xi) + p D_x^2(\xi)) \partial_q - (3f D_x(\xi) + 3q D_x^2(\xi) + p \dot{D}_x^3(\xi)) \partial_f, \quad (3.41)$$

$$V = \eta(x, y) \partial_y + D_x(\eta) \partial_p + D_x^2(\eta) \partial_q + \dot{D}_x^3(\eta) \partial_f, \quad (3.42)$$

The infinitesimal point equivalence transformations (3.41)-(3.42) can be written in the finite form as in (2.17)-(2.18), respectively, where ϕ and ψ are arbitrary functions of the indicated variables.

3.2 Third-order differential invariants and invariant equations under the transformation $\bar{x} = x$, $\bar{y} = \psi(x, y)$

In this section, we derive all the third order differential invariants of the general class $y''' = f(x, y, y', y'')$ under a subgroup of point transformations (2.14), namely the transformations $\bar{x} = x$, $\bar{y} = \psi(x, y)$. Moreover, the invariant differentiation operators [6] are constructed in order to get some higher-order differential invariants from the lower-order ones. Precisely, we obtain the following theorem.

Theorem 3.1. *Let $y''' = f(x, y, y', y'')$ be the class of third-order ODE with $f_{4,4,4} \neq 0$. All the third order differential invariants, under the point transformations $\bar{x} = x$, $\bar{y} = \psi(x, y)$,*

are functions of the following seventeen differential invariants

$$\begin{aligned}
\alpha_1 &= x, & \alpha_2 &= \frac{\lambda_5}{\lambda_4}, & \alpha_3 &= \frac{\lambda_6}{\lambda_4^2}, & \alpha_4 &= \frac{\lambda_7}{\lambda_4^2}, & \alpha_5 &= \frac{\lambda_8}{\lambda_4^2}, & \alpha_6 &= \frac{\lambda_9}{\lambda_4^2}, \\
\alpha_7 &= \frac{\lambda_{10}}{\lambda_4^2}, & \alpha_8 &= \frac{\lambda_{11}}{\lambda_4^2}, & \alpha_9 &= \lambda_{12}, & \alpha_{10} &= \lambda_{13}, & \alpha_{11} &= \frac{\lambda_{14}}{\lambda_4}, & \alpha_{12} &= \frac{\lambda_{15}}{\lambda_4}, \\
\alpha_{13} &= \frac{\lambda_{16}}{\lambda_4}, & \alpha_{14} &= \frac{\lambda_{17}}{\lambda_4}, & \alpha_{15} &= \frac{\lambda_{18}}{\lambda_4}, & \alpha_{16} &= \lambda_{19}, & \alpha_{17} &= \lambda_{20},
\end{aligned} \tag{3.43}$$

where $\{\lambda_i\}_{i=4}^{20}$ are the following relative invariants

$$\begin{aligned}
\lambda_4 &= f_{4,4}, \\
\lambda_5 &= \frac{1}{3}(2f_{3,4}f_4 - 2f_3f_{4,4} - 6f_{2,4} + 3f_{3,3}), \\
\lambda_6 &= f_{4,4,4}, \\
\lambda_7 &= \frac{1}{3}(2f_4f_{4,4,4} + 3f_{3,4,4}), \\
\lambda_8 &= \frac{1}{9}(4f_{4,4,4}f_4^2 + 12f_4f_{3,4,4} + 4f_4f_{4,4}^2 + 9f_{3,3,4} + 6f_{3,4}f_{4,4}), \\
\lambda_9 &= \frac{1}{9}(2f_{4,4,4}f_4^2 + 2f_4f_{4,4}^2 + 3f_4f_{3,4,4} + 3f_{4,4,4}f_3 + 9f_{2,4,4} + 3f_{3,4}f_{4,4}), \\
\lambda_{10} &= \frac{1}{18}(-4f_4^2f_{3,4,4} + 4f_4f_{4,4,4}f_3 + 12f_4f_{2,4,4} - 12f_{3,3,4}f_4 - 4f_4f_{3,4}f_{4,4} - 6f_{3,4}^2 - 9f_{3,3,3} \\
&\quad + 4f_{4,4}^2f_3 + 6f_3f_{3,4,4} + 12f_{2,4}f_{4,4} + 18f_{2,3,4}), \\
\lambda_{11} &= \frac{1}{27}(-4f_4f_3f_{4,4}^2 - 6f_{4,4}f_{3,4}f_3 - 18f_{4,4}f_{2,3} - 12f_{4,4}f_{2,4}f_4 + 4f_{4,4}f_{3,4}f_4^2 - 6f_{4,4,4}f_3^2 - 36f_3f_{2,4,4} \\
&\quad - 4f_3f_{4,4,4}f_4^2 + 9f_3f_{3,3,4} + 12f_{3,4}^2f_4 + 9f_{3,4}f_{3,3} + 4f_4^3f_{3,4,4} - 12f_4^2f_{2,4,4} + 12f_{3,3,4}f_4^2 \\
&\quad + 9f_4f_{3,3,3} - 54f_{2,2,4} + 27f_{2,3,3}), \\
\lambda_{12} &= \frac{1}{3}(-f_4^2 - 3f_3 + 3\dot{D}_xf_4), \\
\lambda_{13} &= \frac{1}{9}(-2f_4^3 - 9f_4f_3 + 6f_4\dot{D}_xf_4 - 27f_2 + 9\dot{D}_xf_3), \\
\lambda_{14} &= \frac{1}{3}(3\dot{D}_xf_{4,4} + f_{4,4}f_4), \\
\lambda_{15} &= \frac{1}{9}(2f_{4,4}f_4^2 + 3f_3f_{4,4} + 6f_4\dot{D}_xf_{4,4} - 9f_{2,4} + 9\dot{D}_xf_{3,4}), \\
\lambda_{16} &= \frac{1}{9}(4f_4^2\dot{D}_xf_{4,4} - 2f_4^2f_{3,4} - 3f_4f_{3,3} + 12f_4\dot{D}_xf_{3,4} + 4f_4f_{4,4}\dot{D}_xf_4 - 12f_{2,4}f_4 + 6f_{3,4}\dot{D}_xf_4 - 18f_{2,3} + 9\dot{D}_xf_{3,3}), \\
\lambda_{17} &= \frac{1}{27}(2f_{4,4}f_4^3 - 6f_4^2f_{3,4} + 6f_4^2\dot{D}_xf_{4,4} - 9f_4f_{3,3} + 12f_4f_{3,4,4} + 9f_4\dot{D}_xf_{3,4} - 9f_{2,4}f_4 + 9f_3\dot{D}_xf_{4,4} - 27f_{2,3} \\
&\quad + 27f_2f_{4,4} + 27\dot{D}_xf_{2,4}), \\
\lambda_{18} &= \frac{1}{27}(-9f_3^2f_{4,4} + 9f_3\dot{D}_xf_{3,4} + 6f_3f_4\dot{D}_xf_{4,4} + 6f_3f_{4,4}\dot{D}_xf_4 + 6f_3f_{3,4}f_4 - 54f_3f_{2,4} + 9f_3f_{3,3} + 4f_{4,4}f_4^2\dot{D}_xf_4 \\
&\quad + 6f_{3,4}f_4\dot{D}_xf_4 + 18f_{2,4}\dot{D}_xf_4 + 27f_2f_{3,4} - 2f_{3,4}f_4^3 - 30f_4^2f_{2,4} + 18f_4f_{4,4}f_2 + 4f_4^3\dot{D}_xf_{4,4} + 12f_4^2\dot{D}_xf_{3,4} \\
&\quad - 3f_4^2f_{3,3} + 9f_4\dot{D}_xf_{3,3} + 18f_4\dot{D}_xf_{2,4} - 45f_4f_{2,3} - 81f_{2,2} + 27\dot{D}_xf_{2,3}), \\
\lambda_{19} &= \frac{1}{9}(-2f_4^3 - 9f_4f_3 - 27f_2 + 9\dot{D}_x^2f_4), \\
\lambda_{20} &= \frac{1}{27}(-2f_4^4 - 12f_4^2f_3 - 6f_4^2\dot{D}_xf_4 + 18f_4\dot{D}_x^2f_4 - 27f_4\dot{D}_xf_3 - 18f_3^2 + 9f_3\dot{D}_xf_4 - 81\dot{D}_xf_2 + 27\dot{D}_x^2f_3),
\end{aligned} \tag{3.44}$$

Moreover, the invariant differential operators are

$$\begin{aligned}
\mathcal{D}_1 &= \frac{1}{f_{4,4}} \tilde{D}_q, \\
\mathcal{D}_2 &= \frac{1}{f_{4,4}} (3\tilde{D}_p + 2f_4 \tilde{D}_q), \\
\mathcal{D}_3 &= \frac{1}{f_{4,4}} (9\tilde{D}_y + 3f_4 \tilde{D}_p + (2f_4^2 + 3f_3) \tilde{D}_q), \\
\mathcal{D}_4 &= \tilde{D}_x + p\tilde{D}_y + q\tilde{D}_p + f\tilde{D}_q.
\end{aligned} \tag{3.45}$$

Proof. Functionally independent solutions of the subsystem

$$X_i J = 0, i = 1 \dots 28, \tag{3.46}$$

of (2.34) provide all independent differential invariants of $y''' = f(x, y, y', y'')$ up to the third order under the transformations $\bar{x} = x$, $\bar{y} = \psi(x, y)$, as well as an implicit solution of the variables K, L, M and N which provide the differential operators via (2.30).

The solution of system (3.46) is found in two steps using Maple through the chain (2.38). First we consider the following subsystem of equations (3.46)

$$X_i J = 0, i = 4 \dots 28. \tag{3.47}$$

In 43-dimensional space of the variables $z_i, i = 1 \dots 43$, the rank of the system (3.47) is

19, so it has 24 functionally independent solutions which are given as:

$$\begin{aligned}
\lambda_1 &= z_1, \\
\lambda_2 &= z_2, \\
\lambda_3 &= z_3, \\
\lambda_4 &= z_{14}, \\
\lambda_5 &= \frac{2}{3} z_{13} z_8 - \frac{2}{3} z_7 z_{14} - 2 z_{11} + z_{12}, \\
\lambda_6 &= z_{24}, \\
\lambda_7 &= \frac{2}{3} z_8 z_{24} + z_{23}, \\
\lambda_8 &= \frac{4}{9} z_{24} z_8^2 + \frac{4}{9} z_8 z_{14}^2 + \frac{4}{3} z_{23} z_8 + z_{22} + \frac{2}{3} z_{13} z_{14}, \\
\lambda_9 &= \frac{2}{9} z_{24} z_8^2 + \frac{1}{3} z_{23} z_8 + \frac{2}{9} z_8 z_{14}^2 + \frac{1}{3} z_{24} z_7 + z_{20} + \frac{1}{3} z_{13} z_{14} \\
\lambda_{10} &= -\frac{2}{9} z_8^2 z_{23} - \frac{2}{3} z_{22} z_8 + \frac{2}{9} z_8 z_{24} z_7 + \frac{2}{3} z_8 z_{20} - \frac{2}{9} z_8 z_{13} z_{14} - \frac{1}{3} z_{13}^2 - \frac{1}{2} z_{21} + \frac{1}{3} z_7 z_{23} + \frac{2}{9} z_{14}^2 z_7 \\
&\quad + \frac{2}{3} z_{14} z_{11} + z_{19}, \\
\lambda_{11} &= \frac{4}{27} z_8^3 z_{23} - \frac{4}{27} z_8^2 z_{24} z_7 + \frac{4}{27} z_8^2 z_{13} z_{14} + \frac{4}{9} z_{22} z_8^2 - \frac{4}{9} z_8^2 z_{20} - \frac{4}{27} z_{14}^2 z_8 z_7 - \frac{4}{9} z_8 z_{14} z_{11} \\
&\quad + \frac{4}{9} z_{13}^2 z_8 + \frac{1}{3} z_{21} z_8 - \frac{2}{9} z_{13} z_7 z_{14} + \frac{1}{3} z_{12} z_{13} - \frac{2}{9} z_{24} z_7^2 + \frac{1}{3} z_7 z_{22} - \frac{4}{3} z_7 z_{20} - 2 z_{17} - \frac{2}{3} z_{14} z_{10} + z_{18}, \\
\lambda_{12} &= -\frac{1}{3} z_8^2 - z_7 + z_{28}, \\
\lambda_{13} &= -\frac{2}{9} z_8^3 - z_7 z_8 + \frac{2}{3} z_8 z_{28} - 3 z_6 + z_{27}, \\
\lambda_{14} &= \frac{1}{3} z_{14} z_8 + z_{34}, \\
\lambda_{15} &= \frac{2}{9} z_{14} z_8^2 + \frac{2}{3} z_8 z_{34} + \frac{1}{3} z_7 z_{14} - z_{11} + z_{33}, \\
\lambda_{16} &= \frac{4}{9} z_8^2 z_{34} - \frac{2}{9} z_8^2 z_{13} + \frac{4}{9} z_8 z_{14} z_{28} + \frac{4}{3} z_8 z_{33} - \frac{4}{3} z_{11} z_8 - \frac{1}{3} z_8 z_{12} + \frac{2}{3} z_{28} z_{13} - 2 z_{10} + z_{32}, \\
\lambda_{17} &= \frac{2}{27} z_{14} z_8^3 + \frac{4}{9} z_{14} z_7 z_8 + z_6 z_{14} - \frac{2}{9} z_8^2 z_{13} + \frac{2}{9} z_8^2 z_{34} + \frac{1}{3} z_8 z_{33} - \frac{1}{3} z_{11} z_8 - \frac{1}{3} z_8 z_{12} + \frac{1}{3} z_7 z_{34} - z_{10} + z_{31}, \\
\lambda_{18} &= \frac{4}{27} z_8^3 z_{34} - \frac{2}{27} z_8^3 z_{13} - \frac{10}{9} z_8^2 z_{11} + \frac{4}{27} z_8^2 z_{14} z_{28} - \frac{1}{9} z_8^2 z_{12} + \frac{4}{9} z_8^2 z_{33} \\
&\quad + \frac{2}{9} z_8 z_{13} z_7 + \frac{2}{3} z_8 z_6 z_{14} + \frac{1}{3} z_8 z_{32} + \frac{2}{3} z_8 z_{31} - \frac{5}{3} z_8 z_{10} + \frac{2}{9} z_8 z_7 z_{34} + \frac{2}{9} z_8 z_{28} z_{13} \\
&\quad + \frac{2}{9} z_{14} z_{28} z_7 - \frac{1}{3} z_{14} z_7^2 + \frac{2}{3} z_{28} z_{11} + z_{30} - 3 z_9 - 2 z_7 z_{11} + z_6 z_{13} + \frac{1}{3} z_7 z_{33} + \frac{1}{3} z_7 z_{12}, \\
\lambda_{19} &= -\frac{2}{9} z_8^3 - z_7 z_8 - 3 z_6 + z_{38}, \\
\lambda_{20} &= -\frac{2}{27} z_8^4 - \frac{4}{9} z_8^2 z_7 - \frac{2}{9} z_8^2 z_{28} - z_8 z_{27} + \frac{2}{3} z_8 z_{38} - \frac{2}{3} z_7^2 + \frac{1}{3} z_{28} z_7 - 3 z_{26} + z_{37},
\end{aligned} \tag{3.48}$$

and

$$\begin{aligned}
\lambda_{21} &= z_{40}, \\
\lambda_{22} &= z_{41}, \\
\lambda_{23} &= \frac{1}{3} z_8 z_{40} z_3 - \frac{1}{3} z_{41} z_8 - z_{40} z_4 + z_{42}, \\
\lambda_{24} &= \frac{2}{3} z_{40} z_8 z_4 + \frac{1}{3} z_{40} z_7 z_3 - z_{40} z_5 - \frac{1}{3} z_7 z_{41} - \frac{2}{3} z_8 z_{42} + z_{43},
\end{aligned} \tag{3.49}$$

In variables $\lambda_i, i = 1 \dots 24$, the remaining non-zero operators in system (3.46) take

the form

$$\begin{aligned}
X_1 &= [0, 1, 0], \\
X_2 &= [0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \lambda_{21}, 0, 0], \\
X_3 &= [0, 0, \lambda_3, -\lambda_4, -\lambda_5, -2\lambda_6, -2\lambda_7, -2\lambda_8, -2\lambda_9, -2\lambda_{10}, \\
&\quad -2\lambda_{11}, 0, 0, -\lambda_{14}, -\lambda_{15}, -\lambda_{16}, -\lambda_{17}, -\lambda_{18}, 0, 0, 0, \lambda_{22}, \lambda_{23}, \lambda_{24}],
\end{aligned} \tag{3.50}$$

Finally, we consider the following subsystem of equations (2.34)

$$X_i J = 0, i = 1 \dots 3. \tag{3.51}$$

In 24-dimensional space of variables $\lambda_i, i = 1 \dots 24$, the rank of the system (3.51) is 3, so it has 21 functionally independent solutions which are given as:

$$\begin{aligned}
\alpha_1 &= x, & \alpha_2 &= \frac{\lambda_5}{\lambda_4}, & \alpha_3 &= \frac{\lambda_6}{\lambda_4^2}, & \alpha_4 &= \frac{\lambda_7}{\lambda_4^2}, & \alpha_5 &= \frac{\lambda_8}{\lambda_4^2}, & \alpha_6 &= \frac{\lambda_9}{\lambda_4^2}, \\
\alpha_7 &= \frac{\lambda_{10}}{\lambda_4^2}, & \alpha_8 &= \frac{\lambda_{11}}{\lambda_4^2}, & \alpha_9 &= \lambda_{12}, & \alpha_{10} &= \lambda_{13}, & \alpha_{11} &= \frac{\lambda_{14}}{\lambda_4}, & \alpha_{12} &= \frac{\lambda_{15}}{\lambda_4}, \\
\alpha_{13} &= \frac{\lambda_{16}}{\lambda_4}, & \alpha_{14} &= \frac{\lambda_{17}}{\lambda_4}, & \alpha_{15} &= \frac{\lambda_{18}}{\lambda_4}, & \alpha_{16} &= \lambda_{19}, & \alpha_{17} &= \lambda_{20},
\end{aligned} \tag{3.52}$$

and

$$\alpha_{18} = \lambda_{21}, \quad \alpha_{19} = -\lambda_4 (\lambda_3 \lambda_{21} - \lambda_{22}), \quad \alpha_{20} = \lambda_4 \lambda_{23}, \quad \alpha_{21} = \lambda_4 \lambda_{24}. \tag{3.53}$$

Here $\alpha_{18}, \alpha_{19}, \alpha_{20}$ and α_{21} are the only invariants depending on the variables K, L, M and N . Then the general solution of (2.31), for $\xi = 0, \eta = \eta(x, y)$, can be given implicitly by back substitution as

$$\begin{aligned}
K &= F_1, \\
f_{4,4}(pK - L) &= F_2, \\
f_{4,4}(f_4(pK - L) - 3(qK - M)) &= F_3, \\
f_{4,4}(f_3(pK - L) + 2f_4(qK - M) - 3(fK - N)) &= F_4.
\end{aligned} \tag{3.54}$$

where F_1, F_2, F_3 and F_4 are the arbitrary functions of $\alpha_i, i = 1 \dots 17$.

Finally, solving system (3.54) gives the variables K, L, M and N in terms of four arbitrary functions F_1, F_2, F_3 and F_4 which provide four independent invariant differentiation operators $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3$ and \mathcal{D}_4 via (2.30). \square

3.3 Third-order differential invariants and invariant equations under the fiber preserving transformation $\bar{x} = \phi(x), \bar{y} = \psi(x, y)$

This section is devoted to the derivation of all third order differential invariants of the general class $y''' = f(x, y, y', y'')$ under a subgroup of point transformations (2.14), namely the fiber preserving transformations $\bar{x} = \phi(x), \bar{y} = \psi(x, y)$. The invariant differentiation operators are also constructed. Precisely, we obtain the following theorem.

Theorem 3.2. *Let $y''' = f(x, y, y', y'')$ be the class of third-order ODE with $f_{4,4,4} \neq 0$. All the third order differential invariants, under pseudo-group of fiber preserving transformations $\bar{x} = \phi(x), \bar{y} = \psi(x, y)$, are functions of the following eleven differential invariants*

$$\begin{aligned} \beta_1 &= \frac{\gamma_6 \gamma_5}{\gamma_4^4}, & \beta_2 &= \frac{\gamma_7 \gamma_5}{\gamma_4^4}, & \beta_3 &= \frac{\gamma_8 \gamma_5^2}{\gamma_4^6}, & \beta_4 &= \frac{\gamma_9 \gamma_5^3}{\gamma_4^8}, & \beta_5 &= \frac{\gamma_5^2 \gamma_{10}}{\gamma_4^4}, & \beta_6 &= \frac{\gamma_5 \gamma_{11}}{\gamma_4^3}, \\ \beta_7 &= \frac{\gamma_{12} \gamma_5^2}{\gamma_4^5}, & \beta_8 &= \frac{\gamma_{13} \gamma_5^3}{\gamma_4^7}, & \beta_9 &= \frac{\gamma_{14} \gamma_5^3}{\gamma_4^7}, & \beta_{10} &= \frac{\gamma_{15} \gamma_5^4}{\gamma_4^9}, & \beta_{11} &= \frac{\gamma_{16} \gamma_5^3}{\gamma_4^6}, \end{aligned} \quad (3.55)$$

where $\{\gamma_i\}_{i=4}^{16}$ are relative invariants given by (3.59).

Moreover, the invariant differential operators are

$$\begin{aligned} \mathcal{D}_1 &= \frac{f_{4,4}}{f_{4,4,4}} \tilde{D}_q, \\ \mathcal{D}_2 &= \frac{1}{f_{4,4} f_{4,4,4}} \left(f_{4,4,4} \tilde{D}_p - f_{3,4,4} \tilde{D}_q \right), \\ \mathcal{D}_3 &= \frac{1}{f_{4,4}^2 f_{4,4,4}} \left(6f_{4,4} f_{4,4,4}^2 \tilde{D}_y - 2f_{4,4} f_{4,4,4} (f_{4,4,4} + 3f_{3,4,4}) \tilde{D}_p + (3f_{4,4} f_{3,4,4}^2 + f_{4,4,4}^2 (3f_{3,3} + 2f_{3,4} f_4 - 6f_{2,4})) \tilde{D}_q \right), \\ \mathcal{D}_4 &= \frac{f_{4,4,4}}{f_{4,4}^2} \left(\tilde{D}_x + p \tilde{D}_y + q \tilde{D}_p + f \tilde{D}_q \right). \end{aligned} \quad (3.56)$$

Proof. Functionally independent solutions of the system (2.34) provide all independent differential invariants of $y''' = f(x, y, y', y'')$ up to the third order under the fiber preserving transformations $\bar{x} = \phi(x), \bar{y} = \psi(x, y)$, as well as an implicit solution of the variables K, L, M and N which provide the differential operators via (2.30).

In variables $\lambda_i, i = 1 \dots 24$, given in (3.44), the remaining non-zero operators in system

(2.34) take the form

$$\begin{aligned}
X_1 &= [0, 1, 0], \\
X_2 &= [0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \lambda_{21}, 0, 0], \\
X_3 &= [0, 0, \lambda_3, -\lambda_4, -\lambda_5, -2\lambda_6, -2\lambda_7, -2\lambda_8, -2\lambda_9, -2\lambda_{10}, \\
&\quad -2\lambda_{11}, 0, 0, -\lambda_{14}, -\lambda_{15}, -\lambda_{16}, -\lambda_{17}, -\lambda_{18}, 0, 0, 0, \lambda_{22}, \lambda_{23}, \lambda_{24}], \\
T_1 &= [1, 0], \\
T_2 &= [0, 0, -\lambda_3, \lambda_4, -\lambda_5, 3\lambda_6, 2\lambda_7, \lambda_8, \lambda_9, 0, -\lambda_{11}, -2\lambda_{12}, -3\lambda_{13}, 0, \\
&\quad -\lambda_{15}, -2\lambda_{16}, -2\lambda_{17}, -3\lambda_{18}, -3\lambda_{19}, -4\lambda_{20}, \lambda_{21}, 0, -\lambda_{23}, -2\lambda_{24}], \\
T_3 &= [0, 0, 0, 0, 0, 0, -\lambda_6, -\frac{2}{3}\lambda_4^2 - 2\lambda_7, -\lambda_7 - \frac{1}{3}\lambda_4^2, \frac{1}{2}\lambda_8 - \lambda_9, 2\lambda_{10} + \frac{1}{3}\lambda_4\lambda_5, 0, -\lambda_{12}, 0, \\
&\quad -\lambda_{14}, -\frac{2}{3}\lambda_4\lambda_{12} - 2\lambda_{15}, \lambda_5 - \lambda_{15}, -\lambda_{16} - \lambda_{17}, -3\lambda_{12}, -2\lambda_{13} - \lambda_{19}, 0, 0, -\lambda_{21}\lambda_3 + \lambda_{22}, \lambda_{23}], \\
T_4 &= [0, 0, 0, 0, \frac{2}{3}\lambda_4, 0, 0, 0, -\frac{1}{3}\lambda_6, -\frac{1}{3}\lambda_7 - \frac{2}{9}\lambda_4^2, -\frac{1}{3}\lambda_8 + \frac{4}{3}\lambda_9, -2, 0, 0, \frac{2}{3}\lambda_4, 0, \\
&\quad -\frac{1}{3}\lambda_{14}, -\frac{2}{9}\lambda_4\lambda_{12} - \frac{1}{3}\lambda_5 - \frac{1}{3}\lambda_{15}, 0, \frac{5}{3}\lambda_{12}, 0, 0, 0, -\frac{1}{3}\lambda_{21}\lambda_3 + \frac{1}{3}\lambda_{22}], \\
T_5 &= [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, -3, 0, 0, 0, 0], \\
T_6 &= [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0].
\end{aligned} \tag{3.57}$$

The solution of system (3.57) is found in two steps using Maple through the chain (2.38).

First we consider the following subsystem of equations (3.57)

$$T_i J = 0, i = 3 \dots 6. \tag{3.58}$$

In 24-dimensional space of variables $\lambda_i, i = 1 \dots 24$, the rank of the system (3.58) is 4, so

it has 20 functionally independent solutions which are given as:

$$\begin{aligned}
\gamma_1 &= \lambda_1, \\
\gamma_2 &= \lambda_2, \\
\gamma_3 &= \lambda_3, \\
\gamma_4 &= \lambda_4, \\
\gamma_5 &= \lambda_6, \\
\gamma_6 &= -\frac{2\lambda_4^2\lambda_7+3\lambda_7^2-3\lambda_6\lambda_8}{3\lambda_6}, \\
\gamma_7 &= -\frac{2\lambda_4^3\lambda_7+3\lambda_7^2\lambda_4-3\lambda_6^2\lambda_5-6\lambda_6\lambda_4\lambda_9}{6\lambda_6\lambda_4}, \\
\gamma_8 &= \frac{2\lambda_6\lambda_4\lambda_5+6\lambda_6\lambda_{10}-6\lambda_7\lambda_9+3\lambda_7\lambda_8}{6\lambda_6}, \\
\gamma_9 &= \frac{4\lambda_6\lambda_4^3\lambda_7\lambda_5+6\lambda_4^2\lambda_6^2\lambda_{11}+12\lambda_4^2\lambda_6\lambda_7\lambda_{10}-6\lambda_4^2\lambda_7^2\lambda_9+3\lambda_4^2\lambda_7^2\lambda_8+3\lambda_6^2\lambda_4\lambda_5\lambda_8-12\lambda_6^2\lambda_4\lambda_5\lambda_9+3\lambda_6\lambda_4\lambda_5\lambda_7^2-3\lambda_6^3\lambda_5^2}{6\lambda_6^2\lambda_4^2}, \\
\gamma_{10} &= \frac{3\lambda_5+\lambda_4\lambda_{12}}{\lambda_4}, \\
\gamma_{11} &= \lambda_{14}, \\
\gamma_{12} &= -\frac{\lambda_{14}\lambda_7+\lambda_6\lambda_5-\lambda_6\lambda_{15}}{\lambda_6}, \\
\gamma_{13} &= -\frac{-3\lambda_{16}\lambda_6^2+6\lambda_6\lambda_7\lambda_{15}+2\lambda_6\lambda_7\lambda_4\lambda_{12}-3\lambda_{14}\lambda_7^2}{3\lambda_6^2}, \\
\gamma_{14} &= \frac{2\lambda_4\lambda_6^2\lambda_{17}+2\lambda_4\lambda_6\lambda_7\lambda_5-2\lambda_4\lambda_6\lambda_7\lambda_{15}+\lambda_4\lambda_{14}\lambda_7^2+\lambda_6^2\lambda_5\lambda_{14}}{2\lambda_6^2\lambda_4}, \\
\gamma_{15} &= \frac{2\lambda_6^3\lambda_4\lambda_5\lambda_{12}+6\lambda_6^3\lambda_4\lambda_{18}+3\lambda_6^3\lambda_5^2+3\lambda_6^3\lambda_5\lambda_{15}}{6\lambda_4\lambda_6^3}, \\
&\quad +\frac{-6\lambda_7\lambda_6^2\lambda_4\lambda_{17}-6\lambda_7\lambda_6^2\lambda_4\lambda_{16}-3\lambda_7\lambda_6^2\lambda_5\lambda_{14}-3\lambda_6\lambda_4\lambda_5\lambda_7^2+9\lambda_7^2\lambda_4\lambda_6\lambda_{15}+2\lambda_7^2\lambda_4^2\lambda_6\lambda_{12}-3\lambda_{14}\lambda_7^3\lambda_4}{6\lambda_4\lambda_6^3}, \\
\gamma_{16} &= -3\lambda_{13} + \lambda_{19}, \\
\gamma_{17} &= \lambda_{21}, \\
\gamma_{18} &= \lambda_{22}, \\
\gamma_{19} &= -\frac{\lambda_7\lambda_{21}\lambda_3-\lambda_7\lambda_{22}-\lambda_{23}\lambda_6}{\lambda_6}, \\
\gamma_{20} &= -\frac{-2\lambda_4\lambda_6^2\lambda_{24}-2\lambda_4\lambda_7\lambda_{23}\lambda_6+\lambda_4\lambda_7^2\lambda_{21}\lambda_3-\lambda_4\lambda_7^2\lambda_{22}-\lambda_6^2\lambda_5\lambda_{21}\lambda_3+\lambda_6^2\lambda_5\lambda_{22}}{2\lambda_6^2\lambda_4},
\end{aligned} \tag{3.59}$$

In variables $\gamma_i, i = 1 \dots 20$, the remaining non-zero operators in system (3.57) take the form

$$\begin{aligned}
X_1 &= [0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0], \\
X_2 &= [0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \gamma_{17}, 0, 0], \\
X_3 &= [0, 0, \gamma_3, -\gamma_4, -2\gamma_5, -2\gamma_6, -2\gamma_7, -2\gamma_8, -2\gamma_9, 0, -\gamma_{11}, -\gamma_{12}, -\gamma_{13}, -\gamma_{14}, -\gamma_{15}, 0, 0, \gamma_{18}, \gamma_{19}, \gamma_{20}], \\
T_1 &= [1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0], \\
T_2 &= [0, 0, -\gamma_3, \gamma_4, 3\gamma_5, \gamma_6, \gamma_7, 0, -\gamma_9, -2\gamma_{10}, 0, -\gamma_{12}, -2\gamma_{13}, -2\gamma_{14}, -3\gamma_{15}, -3\gamma_{16}, \gamma_{17}, 0, -\gamma_{19}, -2\gamma_{20}]
\end{aligned} \tag{3.60}$$

Finally, we consider the following subsystem of equations (2.34)

$$X_i J = 0, i = 1 \dots 3, T_k J = 0, k = 1 \dots 2. \quad (3.61)$$

In 20-dimensional space of variables $\gamma_i, i = 1 \dots 20$, the rank of the system (3.61) is 5, so it has 15 functionally independent solutions which are given as:

$$\begin{aligned} \beta_1 &= \frac{\gamma_6 \gamma_5}{\gamma_4^4}, & \beta_2 &= \frac{\gamma_7 \gamma_5}{\gamma_4^4}, & \beta_3 &= \frac{\gamma_8 \gamma_5^2}{\gamma_4^6}, & \beta_4 &= \frac{\gamma_9 \gamma_5^3}{\gamma_4^8}, & \beta_5 &= \frac{\gamma_5^2 \gamma_{10}}{\gamma_4^4}, & \beta_6 &= \frac{\gamma_5 \gamma_{11}}{\gamma_4^3}, \\ \beta_7 &= \frac{\gamma_{12} \gamma_5^2}{\gamma_4^5}, & \beta_8 &= \frac{\gamma_{13} \gamma_5^3}{\gamma_4^7}, & \beta_9 &= \frac{\gamma_{14} \gamma_5^3}{\gamma_4^7}, & \beta_{10} &= \frac{\gamma_{15} \gamma_5^4}{\gamma_4^9}, & \beta_{11} &= \frac{\gamma_{16} \gamma_5^3}{\gamma_4^6} \end{aligned} \quad (3.62)$$

and

$$\beta_{12} = \frac{\gamma_{17} \gamma_4^2}{\gamma_5}, \quad \beta_{13} = -\frac{(\gamma_{17} \gamma_3 - \gamma_{18}) \gamma_4^3}{\gamma_5}, \quad \beta_{14} = \gamma_{19} \gamma_4, \quad \beta_{15} = \frac{\gamma_5 \gamma_{20}}{\gamma_4}. \quad (3.63)$$

Here $\beta_{12}, \beta_{13}, \beta_{14}$ and β_{15} are the only invariants depending on the variables K, L, M and N . Then the general solution of (2.31), for $\xi = \xi(x), \eta = \eta(x, y)$, can be given implicitly by back substitution as

$$\begin{aligned} \frac{f_{4,4}^2}{f_{4,4,4}} K &= F_1, \\ \frac{f_{4,4}^3}{f_{4,4,4}} (pK - L) &= F_2, \\ \frac{f_{4,4}}{f_{4,4,4}} ((f_4 f_{4,4,4} + 3f_{3,4,4})(pK - L) + 3f_{4,4,4}(qK - M)) &= F_3, \\ \frac{1}{f_{4,4}^2 f_{4,4,4}} ((f_{4,4} f_{3,4,4} (2f_4 f_{4,4,4} + 3f_{3,4,4}) - f_{4,4,4}^2 (2f_{3,4} f_4 - 6f_{2,4} + 3f_{3,3}))(pK - L) \\ + 6f_{4,4} f_{3,4,4} f_{4,4,4} (qK - M) + 6f_{4,4} f_{4,4,4}^2 (fK - N)) &= F_4. \end{aligned} \quad (3.64)$$

where F_1, F_2, F_3 and F_4 are the arbitrary functions of $\beta_i, i = 1 \dots 11$.

Finally, solving system (3.64) gives the variables K, L, M and N in terms of four arbitrary functions F_1, F_2, F_3 and F_4 which provide four independent invariant differentiation operators $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3$ and \mathcal{D}_4 via (2.30). \square

4 Illustrative Examples of Equivalent Equations

We present illustrative examples of third-order ODEs, not quadratic in the second-order derivative, taken from the works [17, 33, 34]. In these studies, the symmetry algebra

properties were investigated in [17], exact solutions for certain classes of third-order ODEs in [33] and a characterization of Lorentzian three-dimensional hyper-CR Einstein-Weyl structures in terms of invariants of the associated third-order ODEs in [34].

Example 4.1. The invariants obtained here may be used when we need to prove the nonequivalence of two given equations under fiber preserving point transformations. Consider the equation

$$\bar{y}''' = \bar{y}''^4, \quad (4.65)$$

with the invariants, given by Theorem 3.2, as

$$\bar{\beta}_1 = 0, \bar{\beta}_2 = 0, \bar{\beta}_3 = 0, \bar{\beta}_4 = 0, \bar{\beta}_5 = \frac{5}{27}, \bar{\beta}_6 = \frac{5}{9}, \bar{\beta}_7 = 0, \bar{\beta}_8 = 0, \bar{\beta}_9 = 0, \bar{\beta}_{10} = 0, \bar{\beta}_{11} = \frac{5}{243}, \quad (4.66)$$

and the equation

$$y''' = y''^3, \quad (4.67)$$

with the invariants, via Theorem 3.2,

$$\beta_1 = 0, \beta_2 = 0, \beta_3 = 0, \beta_4 = 0, \beta_5 = \frac{1}{12}, \beta_6 = \frac{1}{3}, \beta_7 = 0, \beta_8 = 0, \beta_9 = 0, \beta_{10} = 0, \beta_{11} = 0. \quad (4.68)$$

Since their respective invariants are not equal, these two equations (4.65) and (4.67) are obviously nonequivalent with respect to the fiber preserving transformations $\bar{x} = \phi(x)$, $\bar{y} = \psi(x, y)$.

Example 4.2. Consider the equation [33, Section 3.2], also listed in [17, Section 8.3.3],

$$\bar{y}''' = A \bar{y}''^\delta, \quad \delta \neq 0, 1, 2 \quad (4.69)$$

and by setting $\delta = 3$ this becomes

$$\bar{y}''' = A \bar{y}''^3. \quad (4.70)$$

We know the invariants of (4.70) by Theorem 3.2 which are given in (4.68). By means of the fiber preserving transformation

$$\bar{x} = \ln x, \bar{y} = x + y \quad (4.71)$$

this equation (4.70) transforms to

$$x^3 y''' + 3x^2 y'' + xy' + x = A(x + xy' + x^2 y'')^3. \quad (4.72)$$

Equation (4.72) also has the same values of the invariants as given in (4.68).

Now we focus our attention on the third-order ODE

$$\frac{3}{y'^5} y''^2 - \frac{1}{y'^4} y''' = -A \left(\frac{y''}{y'^3} \right)^3 \quad (4.73)$$

This equation (4.73) maps to (4.70) via the interchange transformation

$$\bar{x} = y, \bar{y} = x \quad (4.74)$$

but this transformation is not fiber preserving. The invariants for this ODE (4.73) are not identically constant and so there is no fiber preserving transformation between ODEs (4.73) and (4.70).

Remark 4.3. The special case, of the third order ODE (4.69),

$$\bar{y}''' = \bar{y}''^{\frac{3}{2}} \quad (4.75)$$

defines Einstein-Weyl geometry of hyper-CR type and is of recent interest in physics [34].

It can be characterized by the invariants, by Theorem 3.2, as

$$\bar{\beta}_i = 0, \quad i = 1..11. \quad (4.76)$$

Example 4.4. Consider the equation [33, Section 3.2]

$$\bar{y}''' = A \bar{x}^\alpha \bar{y}''^\delta, \quad \delta \neq 0, 1, 2 \quad (4.77)$$

with the only nonzero third order invariants, by Theorem 3.2, given by

$$\begin{aligned}\bar{\beta}_5 &= \frac{(2\delta-3)(\delta-2)^2}{3\delta(\delta-1)^2} + \frac{\alpha(\delta-2)^2}{A\delta(\delta-1)^2} s, \bar{\beta}_6 = \frac{2(\delta-2)(2\delta-3)}{3\delta(\delta-1)} + \frac{\alpha(\delta-2)}{A\delta(\delta-1)} s, \\ \bar{\beta}_{11} &= \frac{2(\delta-2)^3(\delta-3)(2\delta-3)}{9\delta^2(\delta-1)^3} + \frac{\alpha(\delta-2)^3(\delta-3)}{A\delta^2(\delta-1)^3} s + \frac{\alpha(\alpha-1)(\delta-2)^3}{A^2\delta^2(\delta-1)^3} s^2,\end{aligned}\tag{4.78}$$

where $s = \bar{x}^{(-1-\alpha)}\bar{y}''^{(1-\delta)}$.

It is clear here that the invariants (4.78) are not identically constant. However, their image using a fiber preserving transformation should match the corresponding differential invariants of the transformed equation using the same fiber preserving transformation such that

$$\bar{\beta}_5(\bar{x}, \bar{y}, \bar{y}', \bar{y}'') = \beta_5(x, y, y', y''), \bar{\beta}_6(\bar{x}, \bar{y}, \bar{y}', \bar{y}'') = \beta_6(x, y, y', y''), \bar{\beta}_{11}(\bar{x}, \bar{y}, \bar{y}', \bar{y}'') = \beta_{11}(x, y, y', y'').\tag{4.79}$$

We take $\alpha = 3$, $\delta = 4$ for illustration. Then write this as the transformed third-order ODE

$$\bar{y}''' = A \bar{x}^3 \bar{y}''^4.\tag{4.80}$$

The only nonzero third order invariants of (4.80), by Theorem 3.2, are the following polynomials

$$\bar{\beta}_5 = \frac{5}{27} + \frac{1}{3A} s, \bar{\beta}_6 = \frac{5}{9} + \frac{1}{2A} s, \bar{\beta}_{11} = \frac{5}{243} + \frac{1}{18A} s + \frac{1}{9A^2} s^2,\tag{4.81}$$

where $s = \bar{x}^{-4}\bar{y}''^{-3}$.

By means of the fiber preserving transformation

$$\bar{x} = \frac{1}{x}, \bar{y} = \frac{y}{x}\tag{4.82}$$

this equation (4.80) transforms to

$$y''' = -A x^4 y''^4 - 3 \frac{y''}{x}.\tag{4.83}$$

The only nonzero third order invariants of (4.83), by Theorem 3.2, are

$$\beta_5 = \frac{5}{27} + \frac{1}{3A} \bar{s}, \beta_6 = \frac{5}{9} + \frac{1}{2A} \bar{s}, \beta_{11} = \frac{5}{243} + \frac{1}{18A} \bar{s} + \frac{1}{9A^2} \bar{s}^2, \quad (4.84)$$

where $\bar{s} = x^{-5}y''^{-3}$.

Since the transformation (4.82) transforms the variable s to the variable \bar{s} , then the system (4.79) is satisfied.

Remark 4.5. Using (4.81), it is clear that the variable s is invariant and can be given as $s = A(3\bar{\beta}_5 - \frac{5}{9})$. Therefore, one can use any symbolic package such as Maple or Mathematica to study the equivalence of any ODE $y''' = f(x, y, y', y'')$ to the ODE (4.80) via the fiber preserving transformations $\bar{x} = \phi(x)$, $\bar{y} = \psi(x, y)$. This can be done by comparing the differential invariants (3.55) calculated directly from the ODE $y''' = f(x, y, y', y'')$ and the invariants (4.81) after replacing the variables s by $\bar{s} = A(3\bar{\beta}_5 - \frac{5}{9})$. Moreover, the relation $s = \bar{s}$ may help in constructing the fiber preserving transformation.

Remark 4.6. The invariants (4.81) are not identically constants. However, one can use them to construct identically constant invariants by eliminating the variable s . For examples, the following invariants \bar{I}_1 and \bar{I}_2 are identically zero

$$\bar{I}_1 = 3\bar{\beta}_5 - 2\bar{\beta}_6 + \frac{5}{9} = 0, \bar{I}_2 = \frac{5}{243} + \frac{1}{18} (3\bar{\beta}_5 - \frac{5}{9}) + \frac{1}{9} (3\bar{\beta}_5 - \frac{5}{9})^2 - \bar{\beta}_{11} = 0. \quad (4.85)$$

Example 4.7. Consider the equation [33, Section 3.2]

$$\bar{y}''' = A \bar{y}^{(-7)} \bar{y}'^7 \bar{y}''^3. \quad (4.86)$$

The third order invariants, by Theorem 3.2, are the following polynomials

$$\begin{aligned}
\bar{\beta}_1 &= -\frac{7}{9} s^2 - \frac{7}{18} s, \\
\bar{\beta}_2 &= -\frac{7}{72} s^3 t - \frac{7}{12} s^2 - \frac{7}{24} s, \\
\bar{\beta}_3 &= \frac{49}{432} s^4 t + \frac{7}{27} s^3 + \frac{35}{432} s^2, \\
\bar{\beta}_4 &= \frac{35}{2592} s^6 t^2 + \frac{245}{1296} t s^5 + \left(\frac{245}{648} + \frac{49}{1944} t \right) s^4 + \frac{161}{1296} s^3 + \frac{7}{864} s^2, \\
\bar{\beta}_5 &= \frac{7}{12} s^3 t + \left(\frac{7}{12} - \frac{7}{12} t \right) s^2 + \frac{7}{12} s + \frac{1}{12}, \\
\bar{\beta}_6 &= -\frac{7}{6} t s^2 + \frac{7}{6} s + \frac{1}{3}, \\
\bar{\beta}_7 &= \frac{7}{12} s^3 t - \frac{35}{36} s^2 - \frac{7}{36} s, \\
\bar{\beta}_8 &= -\frac{245}{648} s^4 t + \left(\frac{7}{18} + \frac{49}{216} t \right) s^3 - \frac{35}{216} s^2 - \frac{7}{216} s, \\
\bar{\beta}_9 &= \frac{7}{432} t^2 s^5 + \frac{49}{1296} s^4 t + \left(\frac{49}{108} + \frac{7}{432} t \right) s^3 + \frac{35}{432} s^2, \\
\bar{\beta}_{10} &= -\frac{217}{7776} s^6 t^2 - \frac{343}{2592} t s^5 + \left(-\frac{49}{432} - \frac{245}{3888} t \right) s^4 + \frac{175}{2592} s^3 + \frac{35}{2592} s^2, \\
\bar{\beta}_{11} &= \frac{7}{9} t^2 s^4 - \frac{35}{36} s^3 t,
\end{aligned} \tag{4.87}$$

where $s = \bar{y}^7 \bar{y}'^{(-8)} \bar{y}''^{(-1)}$ and $t = \bar{y}^{(-8)} \bar{y}'^{(10)}$.

By means of the fiber preserving transformation

$$\bar{x} = \frac{x}{x-1}, \quad \bar{y} = \frac{y}{x-1} \tag{4.88}$$

this equation (4.86) transforms to

$$y''' = \frac{y'' \left((y'x - y' - y)^7 (x-1)^{12} y''^2 - 3y^7 \right)}{y^7 (x-1)}. \tag{4.89}$$

The third order invariants calculated directly from the ODE (4.89), by Theorem 3.2, match the invariants (4.87) after replacing the variables s and t by their image $\bar{s} = (x-1)^{(-10)} y^7 (y' - xy' + y)^{(-8)} y''^{(-1)}$ and $\bar{t} = (x-1)^8 y^{(-8)} (y' - xy' + y)^{10}$ under the transformation (4.88).

This completes the examples.

5 Conclusion

We have provided an extension of the work of Bagderina [1] who solved the equivalence problem for scalar third-order ordinary differential equations (ODEs), quadratic in the second-order derivative, via point transformations. Here we considered the equivalence problem for third-order ODEs of the general form $y''' = f(x, y, y', y'')$, which are not quadratic in the second-order derivative, under the pseudo-group of fiber preserving equivalence transformations $\bar{x} = \phi(x)$, $\bar{y} = \psi(x, y)$. We utilized Lie's infinitesimal method to obtain the differential invariants of this general class of ODEs with f not quadratic in y'' . All third order differential invariants of this pseudo-group and the invariant differentiation operators were determined. These are stated as two Theorems 3.1 and 3.2 in Section 3. These yield simple necessary explicit conditions for a third-order ODE to be equivalent to the respective canonical form under pseudo-group of point transformations. As illustrative examples, we gave a number of equations from the existing literature in Section 4.

It would be of further interest to look at the equivalence problem under a more general equivalence group of point transformations. This necessitate further coding in Maple or Mathematica in order to facilitate the large calculations which also have prevailed for the pseudo-group of fiber preserving point transformations considered in this work.

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Appendix A: The differential operators of the homogeneous linear system of PDEs (2.34)

$$\begin{aligned}
X_1 &= [0, 1, 0, \\
&\quad 0, 0, 0, 0, 0, 0] \\
X_2 &= [0, 0, 1, 0, \\
&\quad 0, 0, 0, z_{40}, 0, 0] \\
X_3 &= [0, 0, z_3, z_4, z_5, 0, 0, 0, -z_9, -z_{10}, -z_{11}, -z_{12}, -z_{13}, -z_{14}, -2z_{15}, -2z_{16}, -2z_{17}, -2z_{18}, \\
&\quad -2z_{19}, -2z_{20}, -2z_{21}, -2z_{22}, -2z_{23}, -2z_{24}, z_{25}, 0, 0, 0, -z_{29}, -z_{30}, -z_{31}, -z_{32}, -z_{33}, \\
&\quad -z_{34}, z_{35}, 0, 0, 0, z_{39}, 0, z_{41}, z_{42}, z_{43}] \\
X_4 &= [0, 0, 0, 1, 0, \\
&\quad 0, 0, 0, 0, z_{40}, 0] \\
X_5 &= [0, 0, 0, 2z_3, 3z_4, -z_7, -2z_8, 3, -2z_{10}, -2z_{11} - z_{12}, -z_{13}, -4z_{13}, -2z_{14}, 0, -3z_{16}, -2z_{18} \\
&\quad -2z_{17}, -2z_{19}, -z_{21} - 4z_{19}, -2z_{20} - z_{22}, -z_{23}, -6z_{22}, -4z_{23}, -2z_{24}, 0, 4z_5, -z_{27}, -2z_{28}, \\
&\quad 0, -z_9 - 2z_{30}, -z_{32} - z_{10} - 2z_{31}, -z_{11} - z_{33}, -z_{12} - 4z_{33}, -z_{13} - 2z_{34}, -z_{14}, 5z_{25}, -z_{37}, \\
&\quad -2z_{38}, 0, 6z_{35}, 0, 0, z_{41} + z_{40}z_3, 2z_{42} + z_{40}z_4] \\
X_6 &= [0, 0, 0, z_3^2, 3z_3z_4, z_5 - z_8z_4 - z_7z_3, 3z_4 - 2z_8z_3, 3z_3, z_6 - 2z_3z_{10} - 2z_4z_{11}, -z_4z_{13} - 2z_3z_{11} \\
&\quad -z_3z_{12}, -z_3z_{13} - z_4z_{14}, -2z_8 - 4z_3z_{13}, 3 - 2z_3z_{14}, 0, -3z_3z_{16} - 3z_4z_{17}, -2z_4z_{19} - 2z_3z_{17} \\
&\quad -2z_3z_{18} - z_{10}, -z_{11} - 2z_3z_{19} - 2z_4z_{20}, -z_{12} - z_3z_{21} - 2z_{11} - 4z_3z_{19} - z_4z_{22}, -z_{13} - z_3z_{22} \\
&\quad -2z_3z_{20} - z_4z_{23}, -z_4z_{24} - z_{14} - z_3z_{23}, -6z_{13} - 6z_3z_{22}, -2z_{14} - 4z_3z_{23}, -2z_3z_{24}, 0, 3z_4^2 \\
&\quad + 4z_5z_3, z_{25} - z_5z_8 - z_4z_{28} - z_7z_4 - z_3z_{27}, 3z_5 - 2z_3z_{28} - 2z_8z_4, 3z_4, -z_3z_9 + z_{26} - 2z_4z_{31} \\
&\quad -2z_3z_{30} - 2z_4z_{10} - 2z_5z_{11}, -z_5z_{13} - z_4z_{12} - z_3z_{10} - 2z_3z_{31} - 2z_4z_{11} - z_3z_{32} - z_4z_{33}, -z_4z_{13} \\
&\quad -z_3z_{11} - z_3z_{33} - z_5z_{14} - z_4z_{34}, -2z_{28} - 4z_4z_{13} - z_3z_{12} - 4z_3z_{33}, -2z_3z_{34} - z_3z_{13} - 2z_4z_{14}, \\
&\quad -z_3z_{14}, 10z_5z_4 + 5z_3z_{25}, z_{35} - 2z_5z_{28} - 2z_4z_{27} - z_8z_{25} - z_3z_{37} - z_4z_{38} - z_5z_7, 3z_{25} - 2z_5z_8 \\
&\quad -4z_4z_{28} - 2z_3z_{38}, 3z_5, 10z_5^2 + 6z_3z_{35} + 15z_4z_{25}, 0, 0, z_{41}z_3, z_{41}z_4 + 2z_{42}z_3]
\end{aligned}$$

$$\begin{aligned}
X_7 &= [0, 0, 0, 0, 1, 0, \\
&\quad 0, 0, 0, 0, 0, z_{40}] \\
X_8 &= [0, 0, 0, 0, 3z_3, -z_8, 3, 0, -2z_{11}, -z_{13}, -z_{14}, 0, 0, 0, -3z_{17}, -2z_{19}, -2z_{20}, -z_{22}, -z_{23}, \\
&\quad -z_{24}, 0, 0, 0, 6z_4, -z_{28} - z_7, -2z_8, 3, -2z_{10} - 2z_{31}, -2z_{11} - z_{12} - z_{33}, -z_{13} - z_{34}, \\
&\quad -4z_{13}, -2z_{14}, 0, 10z_5, -2z_{27} - z_{38}, -4z_{28}, 0, 15z_{25}, 0, 0, 0, z_{41} + 2z_{40}z_3] \\
X_9 &= [0, 0, 0, 0, 3z_3^2, 3z_4 - 2z_8z_3, 6z_3, 0, -4z_3z_{11} - z_7, -2z_3z_{13} - 2z_8, 3 - 2z_3z_{14}, 6, 0, \\
&\quad 0, -6z_3z_{17} - 3z_{10}, -4z_3z_{19} - z_{12} - 4z_{11}, -4z_3z_{20} - z_{13}, -2z_3z_{22} - 4z_{13}, -2z_3z_{23} \\
&\quad -2z_{14}, -2z_3z_{24}, 0, 0, 0, 0, 12z_3z_4, -3z_8z_4 - 2z_7z_3 - 2z_3z_{28} + 4z_5, -4z_8z_3 + 9z_4, \\
&\quad 6z_3, -6z_4z_{11} - 4z_3z_{10} - 4z_3z_{31} - z_{27} + z_6, -3z_4z_{13} - 4z_3z_{11} - 2z_3z_{12} - 2z_3z_{33} \\
&\quad -2z_{28}, -3z_4z_{14} - 2z_3z_{13} - 2z_3z_{34}, -8z_3z_{13} - 2z_8, -4z_3z_{14} + 3, 0, 20z_5z_3 + 15z_4^2, \\
&\quad -2z_3z_{38} - 3z_7z_4 - 4z_5z_8 - 4z_3z_{27} - 6z_4z_{28} + 5z_{25}, -6z_8z_4 - 8z_3z_{28} + 12z_5, 9z_4, \\
&\quad 30z_3z_{25} + 60z_5z_4, 0, 0, 0, z_{40}z_3^2 + 2z_{41}z_3] \\
X_{10} &= [0, 0, 0, 0, z_3^3, -z_8z_3^2 + 3z_3z_4, 3z_3^2, 0, -z_8z_4 - 2z_3^2z_{11} - z_7z_3 + z_5, -z_3^2z_{13} - 2z_8z_3 \\
&\quad + 3z_4, 3z_3 - z_3^2z_{14}, 6z_3, 0, 0, -3z_3z_{10} - 3z_4z_{11} - 3z_3^2z_{17} + 2z_6, -2z_3^2z_{19} - 4z_3z_{11} \\
&\quad -z_4z_{13} - z_3z_{12}, -2z_3^2z_{20} - z_3z_{13} - z_4z_{14}, -4z_3z_{13} - z_3^2z_{22} - 2z_8, 3 - 2z_3z_{14} - z_3^2z_{23}, \\
&\quad -z_3^2z_{24}, 6, 0, 0, 0, 6z_3^2z_4, -z_3^2z_{28} + 4z_5z_3 - 3z_8z_3z_4 - z_7z_3^2 + 3z_4^2, -2z_8z_3^2 + 9z_3z_4, \\
&\quad 3z_3^2, -2z_3^2z_{31} - 2z_3^2z_{10} + z_6z_3 - 6z_3z_4z_{11} - z_3z_{27} - z_4z_{28} - z_5z_8 - z_7z_4 + z_{25}, -2z_8z_4 \\
&\quad -2z_3^2z_{11} - 2z_3z_{28} - 3z_3z_4z_{13} - z_3^2z_{33} - z_3^2z_{12} + 3z_5, -z_3^2z_{13} - 3z_3z_4z_{14} - z_3^2z_{34} + 3z_4, \\
&\quad -4z_3^2z_{13} - 2z_8z_3 + 6z_4, 3z_3 - 2z_3^2z_{14}, 0, 15z_3z_4^2 + 10z_5z_3^2, -3z_8z_4^2 - 2z_3^2z_{27} + 10z_5z_4 \\
&\quad + 5z_3z_{25} - 4z_5z_8z_3 - 3z_7z_3z_4 - 6z_3z_4z_{28} - z_3^2z_{38}, -4z_3^2z_{28} + 12z_5z_3 - 6z_8z_3z_4 + 9z_4^2, \\
&\quad 9z_3z_4, 15z_3^2z_{25} + 60z_5z_3z_4 + 15z_4^3, 0, 0, 0, z_{41}z_3^2]
\end{aligned}$$

$$\begin{aligned}
X_{11} &= [0, \\
&\quad 0, 0, 0, 0, 0] \\
X_{12} &= [0, 0, 0, 0, 0, 1, 0, 4 z_3, -z_8, 3, 0, -2 z_{11}, -z_{13}, \\
&\quad -z_{14}, 0, 0, 0, 10 z_4, -z_7 - 2 z_{28}, -2 z_8, 3, 20 z_5, 0, 0, 0, 0] \\
X_{13} &= [0, 0, 0, 0, 0, 3 z_3, 0, 0, -z_8, 3, 0, 0, 0, 0, -3 z_{11}, -z_{13}, -z_{14}, 0, 0, 0, 0, 0, 0, 6 z_3^2, 6 z_4 \\
&\quad -3 z_8 z_3, 9 z_3, 0, -z_7 - z_{28} - 6 z_3 z_{11}, -2 z_8 - 3 z_3 z_{13}, 3 - 3 z_3 z_{14}, 6, 0, 0, 30 z_3 z_4, 10 z_5 \\
&\quad -6 z_3 z_{28} - 3 z_7 z_3 - 6 z_8 z_4, 18 z_4 - 6 z_8 z_3, 9 z_3, 45 z_4^2 + 60 z_5 z_3, 0, 0, 0, 0] \\
X_{14} &= [0, 0, 0, 0, 0, 3 z_3^2, 0, 0, 3 z_4 - 2 z_8 z_3, 6 z_3, 0, 0, 0, 0, -z_7 - 6 z_3 z_{11}, -2 z_3 z_{13} - 2 z_8, \\
&\quad 3 - 2 z_3 z_{14}, 6, 0, 0, 0, 0, 0, 0, 4 z_3^3, 12 z_3 z_4 - 3 z_8 z_3^2, 9 z_3^2, 0, 4 z_5 - 2 z_7 z_3 - 3 z_8 z_4 \\
&\quad -6 z_3^2 z_{11} - 2 z_3 z_{28}, 9 z_4 - 4 z_8 z_3 - 3 z_3^2 z_{13}, 6 z_3 - 3 z_3^2 z_{14}, 12 z_3, 0, 0, 30 z_3^2 z_4, \\
&\quad -12 z_8 z_3 z_4 + 15 z_4^2 + 20 z_5 z_3 - 6 z_3^2 z_{28} - 3 z_7 z_3^2, 36 z_3 z_4 - 6 z_8 z_3^2, 9 z_3^2, 90 z_3 z_4^2 \\
&\quad + 60 z_5 z_3^2, 0, 0, 0, 0] \\
X_{15} &= [0, 0, 0, 0, 0, z_3^3, 0, 0, -z_8 z_3^2 + 3 z_3 z_4, 3 z_3^2, 0, 0, 0, 0, -z_8 z_4 - z_7 z_3 - 3 z_3^2 z_{11} + z_5, \\
&\quad -z_3^2 z_{13} - 2 z_8 z_3 + 3 z_4, 3 z_3 - z_3^2 z_{14}, 6 z_3, 0, 0, 0, 0, 0, 0, z_3^4, 6 z_3^2 z_4 - z_8 z_3^3, 3 z_3^3, \\
&\quad 0, -z_3^2 z_{28} - z_7 z_3^2 - 3 z_8 z_3 z_4 + 3 z_4^2 + 4 z_5 z_3 - 2 z_3^3 z_{11}, -z_3^3 z_{13} + 9 z_3 z_4 - 2 z_8 z_3^2, \\
&\quad -z_3^3 z_{14} + 3 z_3^2, 6 z_3^2, 0, 0, 10 z_3^3 z_4, -z_7 z_3^3 - 6 z_8 z_3^2 z_4 - 2 z_3^3 z_{28} + 10 z_5 z_3^2 + 15 z_3 z_4^2, \\
&\quad 18 z_3^2 z_4 - 2 z_8 z_3^3, 3 z_3^3, 45 z_3^2 z_4^2 + 20 z_5 z_3^3, 0, 0, 0, 0] \\
X_{16} &= [0, 1, 0, 0, \\
&\quad 0, 0, 0, 0, 0, 0] \\
X_{17} &= [0, 1, 0, 0, 0, 0, 0, 0, 0, 5 z_3, \\
&\quad -z_8, 3, 0, 15 z_4, 0, 0, 0, 0] \\
X_{18} &= [0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 4 z_3, 0, 0, -z_8, 3, 0, 0, 0, 0, \\
&\quad 10 z_3^2, 10 z_4 - 4 z_8 z_3, 12 z_3, 0, 60 z_3 z_4, 0, 0, 0, 0]
\end{aligned}$$

$$\begin{aligned}
X_{19} &= [0, 0, 0, 0, 0, 0, 0, 0, 3z_3, 0, 0, 0, 0, 0, -z_8, 3, 0, 0, 0, 0, 0, 0, 0, 0, 0, 6z_3^2, 0, 0, 6z_4 - 3z_8z_3, \\
&\quad 9z_3, 0, 0, 0, 0, 10z_3^3, 30z_3z_4 - 6z_8z_3^2, 18z_3^2, 0, 90z_3^2z_4, 0, 0, 0, 0] \\
X_{20} &= [0, 0, 0, 0, 0, 0, 0, 0, 3z_3^2, 0, 0, 0, 0, 0, 3z_4 - 2z_8z_3, 6z_3, 0, 0, 0, 0, 0, 0, 0, 0, 4z_3^3, 0, 0, \\
&\quad 12z_3z_4 - 3z_8z_3^2, 9z_3^2, 0, 0, 0, 0, 5z_3^4, -4z_8z_3^3 + 30z_3^2z_4, 12z_3^3, 0, 60z_3^3z_4, 0, 0, 0, 0] \\
X_{21} &= [0, 0, 0, 0, 0, 0, 0, 0, z_3^3, 0, 0, 0, 0, 0, -z_8z_3^2 + 3z_3z_4, 3z_3^2, 0, 0, 0, 0, 0, 0, 0, 0, z_3^4, 0, 0, \\
&\quad 6z_3^2z_4 - z_8z_3^3, 3z_3^3, 0, 0, 0, 0, z_3^5, 10z_3^3z_4 - z_8z_3^4, 3z_3^4, 0, 15z_3^4z_4, 0, 0, 0, 0] \\
X_{22} &= [0, \\
&\quad 0, 1, 0, 0, 0, 0] \\
X_{23} &= [0, 1, 0, \\
&\quad 0, 6z_3, 0, 0, 0, 0] \\
X_{24} &= [0, 1, 0, 0, 0, 0, 0, 5z_3, \\
&\quad 0, 0, 15z_3^2, 0, 0, 0, 0] \\
X_{25} &= [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 4z_3, 0, 0, 0, 0, 0, 0, \\
&\quad 10z_3^2, 0, 0, 20z_3^3, 0, 0, 0, 0] \\
X_{26} &= [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 3z_3, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 6z_3^2, 0, 0, 0, 0, 0, 0, \\
&\quad 0, 10z_3^3, 0, 0, 15z_3^4, 0, 0, 0, 0] \\
X_{27} &= [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 3z_3^2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 4z_3^3, 0, 0, 0, 0, 0, 0, \\
&\quad 0, 5z_3^4, 0, 0, 6z_3^5, 0, 0, 0, 0] \\
X_{28} &= [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, z_3^3, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \\
&\quad z_3^5, 0, 0, z_3^6, 0, 0, 0, 0]
\end{aligned}$$

$$\begin{aligned}
T_1 &= [1, 0, \\
&\quad 0, 0, 0, 0, 0, 0] \\
T_2 &= [0, 0, -z_3, -2 z_4, -3 z_5, -3 z_6, -2 z_7, -z_8, -3 z_9, -2 z_{10}, -z_{11}, -z_{12}, 0, z_{14}, -3 z_{15}, \\
&\quad -2 z_{16}, -z_{17}, -z_{18}, 0, z_{20}, 0, z_{22}, 2 z_{23}, 3 z_{24}, -4 z_{25}, -4 z_{26}, -3 z_{27}, -2 z_{28}, -4 z_{29}, \\
&\quad -3 z_{30}, -2 z_{31}, -2 z_{32}, -z_{33}, 0, -5 z_{35}, -5 z_{36}, -4 z_{37}, -3 z_{38}, -6 z_{39}, z_{40}, 0, -z_{42}, -2 z_{43}] \\
T_3 &= [0, 0, 0, -z_3, -3 z_4, 0, z_8, -3, 0, z_{11}, 0, 2 z_{13}, z_{14}, 0, 0, z_{17}, 0, 2 z_{19}, z_{20}, 0, 3 z_{22}, 2 z_{23}, \\
&\quad z_{24}, 0, -6 z_5, -3 z_6, -2 z_7 + z_{28}, -z_8, -3 z_9, -2 z_{10} + z_{31}, -z_{11}, 2 z_{33} - z_{12}, z_{34}, z_{14}, \\
&\quad -10 z_{25}, -7 z_{26}, -5 z_{27} + z_{38}, -3 z_{28}, -15 z_{35}, 0, 0, -z_{40} z_3, -z_{42} - 2 z_{40} z_4] \\
T_4 &= [0, 0, 0, 0, -z_3, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -4 z_4, 0, z_8, -3, 0, z_{11}, \\
&\quad 0, 2 z_{13}, z_{14}, 0, -10 z_5, -3 z_6, -2 z_7 + 2 z_{28}, -z_8, -20 z_{25}, 0, 0, 0, -z_{40} z_3] \\
T_5 &= [0, -z_3, 0, -1, 0, 0, 0, 0, 0, 0, 0, \\
&\quad -5 z_4, 0, z_8, -3, -15 z_5, 0, 0, 0, 0] \\
T_6 &= [0, -z_3, \\
&\quad 0, -1, 0, -6 z_4, 0, 0, 0, 0] \\
T_7 &= [0, \\
&\quad 0, 0, -z_3, 0, 0, 0, 0]
\end{aligned}$$

Appendix B: The nonzero commutators for the Lie algebra \mathcal{L}_{35} of the differential operators of the homogeneous linear system of PDEs (2.34)

$$\begin{aligned}
& [e_2, e_3] = e_2, & [e_2, e_5] = 2e_4, & [e_2, e_6] = e_5, & [e_2, e_8] = 3e_7, & [e_2, e_9] = 2e_8, \\
& [e_2, e_{10}] = e_9, & [e_2, e_{12}] = 4e_{11}, & [e_2, e_{13}] = 3e_{12}, & [e_2, e_{14}] = 2e_{13}, & [e_2, e_{15}] = e_{14}, \\
& [e_2, e_{17}] = 5e_{16}, & [e_2, e_{18}] = 4e_{17}, & [e_2, e_{19}] = 3e_{18}, & [e_2, e_{20}] = 2e_{19}, & [e_2, e_{21}] = e_{20}, \\
& [e_2, e_{23}] = 6e_{22}, & [e_2, e_{24}] = 5e_{23}, & [e_2, e_{25}] = 4e_{24}, & [e_2, e_{26}] = 3e_{25}, & [e_2, e_{27}] = 2e_{26}, \\
& [e_2, e_{28}] = e_{27}, & [e_2, e_{30}] = -e_2, & [e_2, e_{31}] = -e_4, & [e_2, e_{32}] = -e_7, & [e_2, e_{33}] = -e_{11}, \\
& [e_2, e_{34}] = -e_{16}, & [e_2, e_{35}] = -e_{22}, & [e_3, e_4] = -e_4, & [e_3, e_6] = e_6, & [e_3, e_7] = -e_7, \\
& [e_3, e_9] = e_9, & [e_3, e_{10}] = 2e_{10}, & [e_3, e_{11}] = -e_{11}, & [e_3, e_{13}] = e_{13}, & [e_3, e_{14}] = 2e_{14}, \\
& [e_3, e_{15}] = 3e_{15}, & [e_3, e_{16}] = -e_{16}, & [e_3, e_{18}] = e_{18}, & [e_3, e_{19}] = 2e_{19}, & [e_3, e_{20}] = 3e_{20}, \\
& [e_3, e_{21}] = 4e_{21}, & [e_3, e_{22}] = -e_{22}, & [e_3, e_{24}] = e_{24}, & [e_3, e_{25}] = 2e_{25}, & [e_3, e_{26}] = 3e_{26}, \\
& [e_3, e_{27}] = 4e_{27}, & [e_3, e_{28}] = 5e_{28}, & [e_4, e_5] = 3e_7, & [e_4, e_6] = e_8, & [e_4, e_8] = 6e_{11}, \\
& [e_4, e_9] = 3e_{12}, & [e_4, e_{10}] = e_{13}, & [e_4, e_{12}] = 10e_{16}, & [e_4, e_{13}] = 6e_{17}, & [e_4, e_{14}] = 3e_{18}, \\
& [e_4, e_{15}] = e_{19}, & [e_4, e_{17}] = 15e_{22}, & [e_4, e_{18}] = 10e_{23}, & [e_4, e_{19}] = 6e_{24}, & [e_4, e_{20}] = 3e_{25}, \\
& [e_4, e_{21}] = e_{26}, & [e_4, e_{30}] = -2e_4, & [e_4, e_{31}] = -3e_7, & [e_4, e_{32}] = -4e_{11}, & [e_4, e_{33}] = -5e_{16}, \\
& [e_4, e_{34}] = -6e_{22}, & [e_5, e_6] = e_9, & [e_5, e_7] = -4e_{11}, & [e_5, e_9] = 2e_{13}, & [e_5, e_{10}] = 2e_{14}, \\
& [e_5, e_{11}] = -5e_{16}, & [e_5, e_{13}] = 3e_{18}, & [e_5, e_{14}] = 4e_{19}, & [e_5, e_{15}] = 3e_{20}, & [e_5, e_{16}] = -6e_{22}, \\
& [e_5, e_{18}] = 4e_{24}, & [e_5, e_{19}] = 6e_{25}, & [e_5, e_{20}] = 6e_{26}, & [e_5, e_{21}] = 4e_{27}, & [e_5, e_{30}] = -e_5, \\
& [e_5, e_{31}] = -e_8, & [e_5, e_{32}] = -e_{12}, & [e_5, e_{33}] = -e_{17}, & [e_5, e_{34}] = -e_{23}, & [e_6, e_7] = -e_{12}, \\
& [e_6, e_8] = -e_{13}, & [e_6, e_{10}] = 2e_{15}, & [e_6, e_{11}] = -e_{17}, & [e_6, e_{12}] = -e_{18}, & [e_6, e_{14}] = 2e_{20}, \\
& [e_6, e_{15}] = 5e_{21}, & [e_6, e_{16}] = -e_{23}, & [e_6, e_{17}] = -e_{24}, & [e_6, e_{19}] = 2e_{26}, & [e_6, e_{20}] = 5e_{27}, \\
& [e_6, e_{21}] = 9e_{28}, & [e_7, e_8] = 10e_{16}, & [e_7, e_9] = 4e_{17}, & [e_7, e_{10}] = e_{18}, & [e_7, e_{12}] = 20e_{22}, \\
& [e_7, e_{13}] = 10e_{23}, & [e_7, e_{14}] = 4e_{24}, & [e_7, e_{15}] = e_{25}, & [e_7, e_{30}] = -3e_7, & [e_7, e_{31}] = -6e_{11}, \\
& [e_7, e_{32}] = -10e_{16}, & [e_7, e_{33}] = -15e_{22}, & [e_8, e_9] = 3e_{18}, & [e_8, e_{10}] = 2e_{19}, & [e_8, e_{11}] = -15e_{22}, \\
& [e_8, e_{13}] = 6e_{24}, & [e_8, e_{14}] = 6e_{25}, & [e_8, e_{15}] = 3e_{26}, & [e_8, e_{30}] = -2e_8, & [e_8, e_{31}] = -3e_{12}, \\
& [e_8, e_{32}] = -4e_{17}, & [e_8, e_{33}] = -5e_{23}, & [e_9, e_{10}] = 2e_{20}, & [e_9, e_{11}] = -5e_{23}, & [e_9, e_{12}] = -4e_{24}, \\
& [e_9, e_{14}] = 4e_{26}, & [e_9, e_{15}] = 5e_{27}, & [e_9, e_{30}] = -e_9, & [e_9, e_{31}] = -e_{13}, & [e_9, e_{32}] = -e_{18}, \\
& [e_9, e_{33}] = -e_{24}, & [e_{10}, e_{11}] = -e_{24}, & [e_{10}, e_{12}] = -2e_{25}, & [e_{10}, e_{13}] = -2e_{26}, & [e_{10}, e_{15}] = 5e_{28}, \\
& [e_{11}, e_{30}] = -4e_{11}, & [e_{11}, e_{31}] = -10e_{16}, & [e_{11}, e_{32}] = -20e_{22}, & [e_{12}, e_{30}] = -3e_{12}, & [e_{12}, e_{31}] = -6e_{17}, \\
& [e_{12}, e_{32}] = -10e_{23}, & [e_{13}, e_{30}] = -2e_{13}, & [e_{13}, e_{31}] = -3e_{18}, & [e_{13}, e_{32}] = -4e_{24}, & [e_{14}, e_{30}] = -e_{14}, \\
& [e_{14}, e_{31}] = -e_{19}, & [e_{14}, e_{32}] = -e_{25}, & [e_{16}, e_{30}] = -5e_{16}, & [e_{16}, e_{31}] = -15e_{22}, & [e_{17}, e_{30}] = -4e_{17}, \\
& [e_{17}, e_{31}] = -10e_{23}, & [e_{18}, e_{30}] = -3e_{18}, & [e_{18}, e_{31}] = -6e_{24}, & [e_{19}, e_{30}] = -2e_{19}, & [e_{19}, e_{31}] = -3e_{25}, \\
& [e_{20}, e_{30}] = -e_{20}, & [e_{20}, e_{31}] = -e_{26}, & [e_{22}, e_{30}] = -6e_{22}, & [e_{23}, e_{30}] = -5e_{23}, & [e_{24}, e_{30}] = -4e_{24}, \\
& [e_{25}, e_{30}] = -3e_{25}, & [e_{26}, e_{30}] = -2e_{26}, & [e_{27}, e_{30}] = -e_{27}, & [e_{30}, e_{31}] = e_{31}, & [e_{30}, e_{32}] = 2e_{32}, \\
& [e_{30}, e_{33}] = 3e_{33}, & [e_{30}, e_{34}] = 4e_{34}, & [e_{30}, e_{35}] = 5e_{35}, & [e_{31}, e_{32}] = 2e_{33}, & [e_{31}, e_{33}] = 5e_{34}, \\
& [e_{31}, e_{34}] = 9e_{35}, & [e_{32}, e_{33}] = 5e_{35}.
\end{aligned} \tag{5.90}$$

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